1. Discuss the continuity of the function \( f: \mathbb{R} \to \mathbb{R} \) if

(a) \( f(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases} \)

Proof: We need to show that \( \forall \epsilon > 0, \exists \delta > 0 \) such that \( p \in E \) (or \( R \); our metric space) and \( d_x(p, p_0) \delta, \) then \( d_y(f(p), f(p_0)) < \epsilon \). Let \( p_0 = 0 \).

case 1) \( p_0 < 0: d_x(p, 0) = p \) and \( d_y(f(p), f(0)) = d_y(0, 0) = 0 \)

case 2) \( p_0 \leq 0: d_x(p, 0) = p \) and \( d_y(f(p), f(0)) = d_y(p, 0) = p \).

Choose \( \forall \epsilon > 0, \exists d_x(p, 0) = p < \delta \) and \( d_y(p, 0) = p < \epsilon \).

(b) \( f(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \)

Proof: We need to check if \( \lim_{x \to 0} f(x) = f(0) \) so, \( \lim_{x \to 0} x \sin \frac{1}{x} = 0 \).

case 1) If \( x \) is nonzero, \( 0 \leq \sin \frac{1}{x} \leq 1 \), and we have \( 0 \leq \left| x \sin \frac{1}{x} \right| \leq |x| \) for all nonzero \( x \).

by squeeze theorem, \( \lim_{x \to 0} 0 \leq \lim_{x \to 0} \left| x \sin \frac{1}{x} \right| \leq \lim_{x \to 0} |x| \), then \( 0 \leq \lim_{x \to 0} \left| x \sin \frac{1}{x} \right| \leq 0 \).

Thus, \( \lim_{x \to 0} \left| x \sin \frac{1}{x} \right| = 0 \).

case 2) If \( x \) is zero, \( \lim_{x \to 0} 0 = 0 \).

Therefore, it is continuous.

Discussion:
\( f(x) = x^2 \) and \( f(x) = 1 \) are continuous functions.

Select \( p = 0, q \neq 0: d(p = 0, q) = q \) and \( d(f(p = 0), f(q)) = d(1, q^2) = |1 - q^2| \). The inequality \( |1 - q^2| < \epsilon \).

Test with \( \epsilon = \frac{1}{2}: \sqrt{\frac{1}{2} - \frac{1}{2}} < q < \sqrt{\frac{1}{2} + \frac{1}{2}} \Rightarrow \sqrt{\frac{1}{2} - \frac{1}{2}} < q < \sqrt{\frac{1}{2} + \frac{1}{2}} \). But we require a \( \delta \) such that \( q < \delta \) and \( q < \sqrt{\frac{1}{2}} \). No selection of \( \delta \) will satisfy both inequalities, so the function must be discontinuous at zero.

(d) Claim: \( f \) is not continuous

\( f(x) = \begin{cases} 0 & \text{if } x \text{ is not rational} \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ where } p \text{ and } q \text{ are integers with no common divisors other than } \pm 1, \text{ and } q > 0 \end{cases} \)

Suppose that we want to check the continuity at some rational number \( \frac{p}{q} = x_0 \). Then \( \forall \epsilon > 0 \exists \delta > 0 \) s.t. \( |x - x_0| < \delta \) implies \( |f(x) - f(x_0)| < \epsilon \). Let \( x \) be an irrational number such that \( |x - x_0| < \delta \). This is possible by LUB 5 on page 26 in the text. Now given \( \epsilon < \frac{1}{q} \), then we have \( \forall \epsilon > 0 |x - x_0| < \delta \) but \( |f(x) - f(x_0)| = \left| 0 - \frac{1}{q} \right| = \frac{1}{q} \geq \epsilon \) \( \forall \delta \) and irrational \( x \).
Let $E, E'$ be metric spaces, $f : E \to E'$ a continuous function. Show that if $S$ is a closed subset of $E'$ then $f^{-1}(S)$ is a closed subset of $E$. Derive from this the results that if $f$ is a continuous real-valued function on $E$ then the sets $\{ p \in E : f(p) \leq 0 \}$, $\{ p \in E : f(p) \geq 0 \}$, $\{ p \in E : f(p) = 0 \}$ are closed.

Solution:

Need to Show: $C(f^{-1}(S))$ is an open subset of $E$.

$S \subset E'$ closed $\implies C(S) \subset E'$ open. For continuous functions, the inverse image of an open subset is open, so $f^{-1}(C(S)) \subset E$ is open. We desire to move the complement outside the inverse mapping. This is possible because $C(f(S)) = \{ p \in E : f(p) \in C(S) \} = \{ p \in E : f(p) \notin S \} = C(f^{-1}(S))$.

Now we have that $(f^{-1}(S))$ is closed. So the mapping $f : E \to \mathbb{R}$ maps closed subsets to closed subsets. Then $\{ p \in E : f(p) \leq 0 \}$, $\{ p \in E : f(p) \geq 0 \}$, $\{ p \in E : f(p) = 0 \}$ are closed subsets because they map to the closed subsets $\{ x \in \mathbb{R} : x \leq 0 \}$, $\{ x \in \mathbb{R} : x \geq 0 \}$, $\{ 0 \}$.

9.a. Given some $x_1 \geq 0$ and some $\epsilon > 0$, we seek to find a $\delta > 0$ such that $|x_1 - x_2| < \delta$ implies that $|\sqrt{x_1} - \sqrt{x_2}| < \epsilon$. Here, note that regardless of which is greater, we can say that

$$\sqrt{x_1} - \sqrt{x_2} = (\sqrt{x_1} - \sqrt{x_2}) \left( \frac{\sqrt{x_1} + \sqrt{x_2}}{\sqrt{x_1} + \sqrt{x_2}} \right)$$

for $\{x_1, x_2\} \neq \{0, 0\}$

First let us examine the case that $x_1 = 0$. Here, $|\sqrt{x_1} - \sqrt{x_2}| = |\sqrt{x_2}|$. Here, choosing $\delta < \epsilon^2$ gives us $\sqrt{x}$ is continuous.

Now let us examine the case that $x_1 > 0$. Using the fact that

$$\sqrt{x_1} - \sqrt{x_2} = (\sqrt{x_1} - \sqrt{x_2}) \left( \frac{\sqrt{x_1} + \sqrt{x_2}}{\sqrt{x_1} + \sqrt{x_2}} \right)$$

and distributing, we see that

$$\sqrt{x_1} - \sqrt{x_2} = \frac{x_1 - x_2}{\sqrt{x_1} + \sqrt{x_2}}$$

Since $\sqrt{x_1} > 0$ and $\sqrt{x_2} > 0$, we know that $\sqrt{x_1} + \sqrt{x_2} > \sqrt{x_1}$ which lets us say that

$$\frac{x_1 - x_2}{\sqrt{x_1} + \sqrt{x_2}} < \frac{x_1 - x_2}{\sqrt{x_1}}$$

Here, if we select $\delta = \sqrt{x_1} * \epsilon$, we see that $x_1 - x_2 < \sqrt{x_1} * \epsilon$ so that

$$\frac{x_1 - x_2}{\sqrt{x_1} + \sqrt{x_2}} < \frac{x_1 - x_2}{\sqrt{x_1}} = \frac{\sqrt{x_1} + \epsilon}{\sqrt{x_1}} = \epsilon$$

Thus given any $x_1 \in \mathbb{R}$ and any $\epsilon > 0$, we see that $\exists \delta > 0$ such that $d(x_1, x_2) < \delta \implies d(\sqrt{x_1}, \sqrt{x_2}) < \epsilon$ and thus that $\sqrt{x}$ is continuous $\forall x \in \mathbb{R}$.

9.b. Here we note that

$$x - 1 = (\sqrt{x} - 1)(\sqrt{x} + 1)$$

which lets us say that

$$\frac{x - 1}{\sqrt{x}} = \frac{(\sqrt{x} - 1)(\sqrt{x} + 1)}{\sqrt{x} - 1} = \sqrt{x} + 1$$

From this we see that

$$\lim_{x \to 1} \frac{x - 1}{\sqrt{x} - 1} = \lim_{x \to 1} \sqrt{x} + 1 = 2$$
13. Write down the details of the following alternate proof that a continuous real valued function \( f \) on a compact metric space \( E \) is bounded and attains a maximum:

If \( f \) is not bounded, then for \( n = 1, 2, 3 \ldots \) there is a point \( p_n \in E \) such that \( |f(p_n)| > n \).

and a contradiction arises from the existence of a convergent subsequence of \( p_1, p_2, p_3, \ldots \)

Thus \( f \) is bounded and we can find a sequence of points \( q_1, q_2, q_3, \ldots \) of \( E \)

such that \( \lim_{n \to \infty} f(p_n) = \text{l.u.b.} \{f(p) : p \in E\} \).

A maximum will be attained by \( f \) at the limit of a convergent subsequence of \( q_1, q_2, q_3, \ldots \).

Proof: Assume that \( f \) is not bounded, then for \( n = 1, 2, 3 \ldots \)

there is a point \( p_n \in E \) such that \( |f(p_n)| > n \).

Since \( E \) is compact, \( \exists \) a convergent subsequence \( \{p_{n_k}\} \) with \( \lim_{k \to \infty} p_{n_k} = p_0, p_0 \in E \).

Now because \( f \) is continuous, we can use the property of continuity, \( \lim_{k \to \infty} f(p_{n_k}) = f(p_0) \).

Then we could find a positive integer \( N \) such that

\[
1 > |f(p_{n_k}) - f(p_0)| = |f(p_{n_k})| - |f(p_0)| > n_k - |f(p_0)|
\]

\( \Rightarrow 1 + |f(p_0)| > n_k \) but we cannot which is contradiction.

Therefore \( f \) is bounded and nonempty. It implies that we can find a sequence in \( f(E) \) where

\( \lim_{n \to \infty} f(q_n) = \text{l.u.b.} \{f(p) : p \in E\} \).

Since \( E \) is compact, \( \exists \) a convergent subsequence \( \{q_{n_k}\} \) of \( \{q_n\} \) where \( \lim_{n \to \infty} q_{n_k} = q_0, q_0 \in E \).

However, since \( \lim_{n \to \infty} f(q_n) = \lim_{k \to \infty} f(q_{n_k}) \), we must have \( f(q_0) = \text{l.u.b.} \{f(p) : p \in E\} \).

Thus, \( f(q_0) \geq f(p), \forall p \in E, f(q_0) \) is max.
14. (a) Prove that if $S$ is a nonempty compact subset of a metric space $E$ and $p_0 \in E$ then $\min\{d(p_0, p): p \in S\}$ exists (distance from $p_0$ to $S$).

Proof: Let $f(x) = d(p, p_0) = |p - p_0|$ where $f: E \to \mathbb{R}$.

This is continuous function since if we choose $\delta = \epsilon$,

$$|f(p) - f(p_1)| = |d(p, p_0) - d(p_1, p_0)| \leq d(p, p_1) < \epsilon$$

so $d(p, p_1) < \epsilon$ then $|f(p) - f(p_1)| < \epsilon$.

also, if $f: E \to E$ is continuous and $S$ is a subset of $E$,

then the restriction of $f$ to $S$ is continuous on $S$, too. Therefore, $f(x)$ is continuous on $S$.

Since $f$ is a continuous real valued function on a nonempty compact space $S$,

$f$ attains a minimum at some point by Corollary 2 on textbook p78.

(b) Prove that if $S$ is a nonempty closed subset of $E^n$ and $p_0 \in E^n$ then $\min\{d(p_0, p): p \in S\}$ exists.

Proof:

If $p_0 \in E$ then $\min\{d(p_0, p): p \in S\} = 0$, we can assume that $p_0 \in S^c$. Choose $\epsilon > 0$,

such that the closed ball $B(p_0, \epsilon) \subset E^n$ contains points in $S$, or $B(p_0, \epsilon) \cap S = \emptyset$.

Consider the continuous function $f : E^n \to \mathbb{R}$ given by the function $f(p) = d(p_0, p)$.

Then $f$ is continuous on $\overline{B(p_0, \epsilon)} \cap S$ as well.

Observe that $\overline{B(p_0, \epsilon)} \cap S$ is closed and bounded subset of $E^n$, and it is compact.

by Corollary 2 on textbook p78, $f$ attains a minimum at some point in $\overline{B(p_0, \epsilon)} \cap S$ or $\min\{d(p_0, p): p \in S\}$
15. Prove that for any nonempty compact metric space $E$, $\max\{d(p, q) : p, q \in E\}$ exists.

**Proof:**
Since $E$ is compact and $\{d(p, q) : p, q \in E\}$ is bounded and nonempty, $E$ must be bounded. Then we can find a sequence of points $\{(p_n, q_n)\}_{n=1, 2, \ldots}$ of $E$ s.t.

$$\lim_{n \to \infty} d(p_n, q_n) = \sup\{d(p, q) : p, q \in E\}.$$ 

Since $E$ is compact, $\exists$ convergent subsequences $\{p_n\}, \{q_n\}$ where $\{p_n\}, \{q_n\}$ converge to some point $p_0, q_0 \in E$. Thus, we have that

$$d(p_0, q_0) = \lim_{k \to \infty} d(p_{n_k}, q_{n_k}) = \lim_{n \to \infty} d(p_n, q_n) = \sup\{d(p, q) : p, q \in E\}.$$

Hence, $\max\{d(p, q) : p, q \in E\}$ exists. □

10. Discuss the continuity of the function $f : E^2 \to \mathbb{R}$ if

(a) $f(x, y) = \begin{cases} \frac{1}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

(b) $f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

(c) $f(x, y) = \begin{cases} \frac{x^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

**Proof:**

(a) By Lemma and Proposition on pg.75-76 in section 4.3, we know that $f(x, y)$ is continuous when $(x, y) \neq (0, 0)$. NTS (rather check) continuity at $(0, 0)$. Then by the last Proposition in section 4.2 on pg. 74, which was also used on prob.2, if $f(x, y)$ is continuous at $(0, 0)$, then $\forall \{p_n\}_{n \geq 1}$ converges to $(0, 0)$ we will have $\lim_{n \to \infty} f(p_n) = f(0, 0) = 0$. Thus, choose $p_n = (\frac{1}{n}, \frac{1}{n})$, so then we have that $\lim_{n \to \infty} p_n = (0, 0)$ and

$$f(p_n) = \frac{1}{(\frac{1}{n})^2 + (\frac{1}{n})^2} = 2n^2.$$ 

So we see that as $n \to \infty$, we have $f(p_n) \to \infty$, which is a contradiction. Hence $f(x, y)$ is not continuous at $(0, 0)$. □

(b) Using the same idea as (a), NTS continuity at $(0, 0)$. Choose $\{p_n\}_{n \geq 1}$ converges to $(0, 0)$ as $p_n = (\frac{1}{n}, \frac{1}{n})$. Then

$$\lim_{n \to \infty} f(p_n) = \lim_{n \to \infty} \frac{1}{\frac{1}{n}^2 + (\frac{1}{n})^2} = \frac{1}{2} \neq 0 = f(0, 0).$$

A contradiction, hence $f(x, y)$ is not continuous at $(0, 0)$. □

(c) NTS continuity at $(0, 0)$. By definition of continuity, $\forall \epsilon > 0$, if $\exists \delta > 0$, s.t. $\forall p = (x_p, y_p)$ $d(p, (0, 0)) = \sqrt{x_p^2 + y_p^2} < \delta$ where $d(f(p), 0) = |f(p)| < \epsilon$.

$f(x, y)$ is continuous at $(0, 0)$. Check.

$$|f(p)| = \left| \frac{x_p y_p^2}{x_p^2 + y_p^2} \right| \leq \left| \frac{x_p y_p^2}{2x_p y_p} \right| = \frac{|y_p|}{2} < \epsilon \iff |y_p| < 2\epsilon.$$

Since $x_p^2 \geq 0$, let $\delta = 2\epsilon$, then for $\sqrt{x_p^2 + y_p^2} < \delta$ $|y_p| < 2\epsilon$ $|f(p)| < \epsilon$. Hence $f(x, y)$ is continuous at $(0, 0) \implies f(x, y)$ is continuous on $E^2$. □