17. Is the function $x^2$ uniformly continuous on $\mathbb{R}$? The function $\sqrt{|x|}$? Why?

Let us show that $x^2$ is not uniformly continuous by showing that $\forall \epsilon > 0 \ \forall \delta > 0 \ \exists x, y \in \mathbb{R}$ such that $|x - y| < \delta$ and $|x^2 - y^2| \geq \epsilon$. To do this, choose $x = \frac{\delta}{4}$ and set $y = \frac{\delta}{2} + \frac{\delta}{2}$. We see that $|x - y| = \frac{\delta}{2} + \frac{\delta}{2} - \frac{\delta}{2} = \frac{\delta}{2}$. However, we also see that $|x^2 - y^2| = \left(\frac{\delta}{4} + \frac{\delta}{2}\right)^2 - \left(\frac{\delta}{4}\right)^2 = \frac{\delta^2}{16} + \epsilon + \frac{\delta}{4} - \frac{\delta^2}{16} = \epsilon + \frac{\delta^2}{4} = \epsilon + \frac{\delta^2}{4} > \epsilon$. Thus $\forall \epsilon > 0 \ \forall \delta > 0 \ \exists x, y \in \mathbb{R}$ such that $|x - y| < \delta$ and $|x^2 - y^2| \geq \epsilon$ and the function $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.

Since $f(x) = \sqrt{|x|}$ is the composition of two functions (namely $\sqrt{x}$ and $|x|$) we can show that each of these functions is uniformly continuous and use the result from question $20$ to conclude that $f$ is also uniformly continuous. First, let us show that $\sqrt{x}$ is uniformly continuous on $[0, \infty)$. To do this, we must find a $\delta > 0$ which is a function of $\epsilon$ only such that $|x - y| < \delta$ implies that $\sqrt{x} - \sqrt{y} < \epsilon$. Let us choose $\delta = \epsilon^2$. Then $|x - y| < \delta$ implies that $|x - y| < \delta$. Now there are two possibilities here: either that $x - y < 0$ or $x - y > 0$. Say $x - y < 0$ so that $|x - y| = y - x$. Then we have $y - x < \epsilon^2$ and thus that $y < x + \epsilon^2$. Since $y > 0$ and $x + \epsilon^2 > 0$, we can take the square root of both sides while still preserving the inequality, giving us $\sqrt{y} < \sqrt{x} + \epsilon^2$. By definition, for any $u, v > 0$ we know that $\sqrt{u} + \sqrt{v} < \sqrt{u} + \sqrt{v}$. Here, this lets us say that $\sqrt{y} < \sqrt{x} + \epsilon^2$ implies that $\sqrt{y} < \sqrt{x} + \epsilon$. Subtracting $\sqrt{x}$ from both sides then gives $\sqrt{y} - \sqrt{x} < \epsilon$ whenever $|x - y| < \epsilon$.

Now let us consider the case that $x - y > 0$. Then $|x - y| = x - y > 0$ and we have $0 < x - y < \delta = \epsilon^2$. Adding $y$ gives $y < x + \epsilon^2$. Taking the square root gives $\sqrt{y} < \sqrt{x} + \epsilon^2$. Again using the aforementioned property of inequalities containing square roots, we may now say that $\sqrt{y} < \sqrt{x} + \epsilon$. Subtracting $\sqrt{x}$ then gives $0 < \sqrt{x} - \sqrt{y} < \epsilon$. Combining this inequality with the one from end the of the preceding paragraph gives us $|\sqrt{x} - \sqrt{y}| < \epsilon$ whenever $|x - y| < \delta$ for every $x, y \in [0, \infty)$. Thus $\sqrt{x}$ is uniformly continuous.

For the function $|x|$ we can show uniform continuity by selecting $\delta = \epsilon$. Here, $|x - y| < \delta$ implies $|x - y| < \epsilon$ and, using the reverse triangle inequality we see that $|f(x) - f(y)| = ||x| - |y|| \leq |x - y| < \epsilon$ thus giving us $|x| - |y| < \epsilon$. Thus $\sqrt{x}$ and $|x|$ are both uniformly continuous and their composition is as well.

18. Prove that for any metric space $E$, the identity function on $E$ is uniformly continuous.

The identity function is defined as $Id(x) = x$, $\forall x \in E$. Here we need to show that given any $\epsilon > 0$, $\exists \delta > 0$ such that $\forall x, y \in E, d(x, y) < \delta$ implies that $d(Id(x), Id(y)) < \epsilon$. Given some $\epsilon > 0$, let us set $\delta = \epsilon$. Then, taking any $x, y \in E$ such that $d(x, y) < \delta$, using $\delta = \epsilon$ we can say that $d(x, y) < \epsilon$. Now, since $Id(x) = x$ and $Id(y) = y$, we may substitute these into the preceding inequality, giving us $d(Id(x), Id(y)) < \epsilon$. Thus we have shown that $d(x, y) < \delta$ implies that $d(Id(x), Id(y)) < \epsilon$ and the proof is complete.
19. Prove that for any metric space $E$ and any $p_0 \in E$, the real-valued function sending any $p$ into $d(p_0, p)$ is uniformly continuous.

Let $f(p) = d(p_0, p)$ where $p_0$ is any point in $E$. Here we wish to show that $f$ is uniformly continuous on $E$. To do this, we must show that given any $\epsilon > 0$, $\exists \delta > 0$ such that $\forall q, r \in E$, $d(q, r) < \delta$ implies that $d(f(q), f(r)) < \epsilon$. Here, note that, via the triangle inequality, $d(q, r) \geq |d(q, p_0) - d(p_0, r)|$. This means that $\delta > d(q, r)$ implies that $\delta > |d(q, p_0) - d(p_0, r)|$. Since $f(q) = d(q, p_0)$ and $f(r) = d(r, p_0)$, we see that the preceding statement implies that $\delta > |f(q) - f(r)|$. Thus here, given any $\epsilon > 0$, setting $\delta = \epsilon$ gives us that $d(q, r) < \delta$ implies $d(f(q), f(r)) < \epsilon$ for every $q, r \in E$ and the proof for uniform continuity is complete.

20. State precisely and prove: A uniformly continuous function of a uniformly continuous function is uniformly continuous.

Here let $f : E \to E'$ and $g : E' \to S$. We seek to show that $h = g \circ f : E \to S$ is uniformly continuous if $f$ and $g$ are. To do this, we must show that given any $\epsilon_h > 0$ $\exists \delta_h > 0$ such that $d_E(p, q) < \delta_h$ implies that $d_S(h(p), h(q)) < \epsilon_h$, $\forall p, q \in E$.

Now, since $g$ is uniformly continuous, given some $\epsilon_g > 0$ we know that $\exists \delta_g$ such that $\forall r, s \in E'$, $d_{E'}(r, s) < \delta_g$ implies that $d_S(g(r), g(s)) < \epsilon_g$. Using the fact that $f$ is uniformly continuous, we can set $\epsilon_f = \delta_g$ and we know that $\forall u, v \in E$ we have $d_E(u, v) < \delta_f$ implies that $d_{E'}(f(u), f(v)) < \epsilon_f = \delta_g$ which in turn implies that $d_S(g(f(u)), g(f(v))) < \epsilon_h$. Rewriting this last inequality, we have $d_S((g \circ f)(u), (g \circ f)(v)) < \epsilon_h$ whenever $d_E(u, v) < \delta_f$. Thus given any $\epsilon_h > 0$ we can set $\delta_h = \delta_f$ and be guaranteed that $d_E(u, v) < \delta_f$ implies that $d_S(h(u), h(v)) < \epsilon_h$ $\forall u, v \in E$, meaning that $h : E \to S$ is uniformly continuous and the proof is complete.
Problem 22

Assume the norm on $V$ is $\| \cdot \|_1$, and the one on $V'$ is $\| \cdot \|_2$.

a. Assume $f$ is continuous at a point $x_0$.

Let $\varepsilon > 0$. There exists $\delta > 0$ such that

$$\forall \ z \in V, \ \| z - x_0 \|_1 < \delta \Rightarrow \| f(z) - f(x_0) \|_2 < \varepsilon$$

Assume $\| x - y \|_1 < \delta$ for some $x, y \in V$.

One has $\| x - y \|_1 = \| x - y + x_0 - x_0 \|_1$.

Thus $\| x - y + x_0 - x_0 \|_1 < \delta$ and

$$\| f(x - y + x_0) - f(x_0) \|_2 < \varepsilon$$

That is $\| f(x) - f(y) + f(x_0) - f(x_0) \|_2 < \varepsilon$ and so

$$\| f(x) - f(y) \|_2 < \varepsilon$$

Hence $\| x - y \|_1 < \delta \Rightarrow \| f(x) - f(y) \|_2 < \varepsilon$.

So $f$ is uniformly continuous and then continuous everywhere.
b. Assume that \( \| f(x) \| / \| x \| \), \( x \in V, x \neq 0 \) above is bounded say by \( M > 0 \).

For \( x \neq 0 \), \( \| f(x) \| = \| x \| \frac{\| f(x) \|}{\| x \|} \leq M \| x \| \).

For \( x = 0 \) we also have \( \| f(x) \| = 0 \leq M \| x \| = 0 \).

Hence \( \forall x \in V, \| f(x) \| \leq M \| x \| \).

Let \( \varepsilon > 0 \). Take \( \delta = \frac{\varepsilon}{M} \).

\( \| x \| < \delta \Rightarrow \| x \| < \frac{\varepsilon}{M} \Rightarrow \| f(x) \| < \varepsilon \).

Hence \( f \) is continuous at zero and therefore is continuous everywhere.

\( \Rightarrow \) Assume \( f \) is continuous. Let us show that \( A \) is bounded.

Since \( f \) is continuous at \( 0 \), \( \forall \varepsilon > 0 \), \( \exists \delta > 0 \) such that \( \| x \| < \delta \Rightarrow \| f(x) \| < \varepsilon \).

For \( x \neq 0 \), \( x = \frac{2 \| x \|}{\delta} \cdot \delta \cdot \frac{1}{2 \| x \|} \cdot x \).

Since \( \| \frac{\delta}{2 \| x \|} x \| = \frac{\delta}{2} < \delta \), \( \| f \left( \frac{\delta}{2 \| x \|} x \right) \| < \varepsilon \).

Thus \( \frac{\| f(x) \|}{\| x \|} = \frac{2 \| x \|}{\delta} \cdot \frac{\| f \left( \frac{\delta}{2 \| x \|} x \right) \|}{\| x \|} < \frac{2 \| x \|}{\delta} \cdot \frac{\varepsilon}{2} \cdot \| x \| \).

So \( f \) is bounded below by \( 0 \) and above by \( \frac{2 \varepsilon}{\delta} \).
C. Let \((v_1, \ldots, v_n)\) be a basis of \(V\). Consider the function \(\| \cdot \| : \mathbb{R}^n \to \mathbb{R}
\)
\[(x_1, \ldots, x_n) \mapsto \| \sum x_i v_i \|\]
\(\| \cdot \|\) is a norm on \(\mathbb{R}^n\). Indeed:
\[\| (x_1, \ldots, x_n) \| = 0 \implies \sum x_i v_i = 0 \implies \sum x_i = 0 \implies x_i = 0, \quad i = 1, \ldots, n\]
\[\| x \| = 0 \implies x = 0.\]
\[\| x + y \| = \| \sum (x_i + y_i) v_i \| = \| \sum x_i v_i + \sum y_i v_i \| \leq \| \sum x_i v_i \| + \| \sum y_i v_i \| = \| x \| + \| y \|.\]
\[\| d x \| = \| \sum \lambda x_i v_i \| = |d| \| \sum x_i v_i \| = |d| \| x \|.\]

Since \(\| \cdot \|\) is a norm it is continuous on \(\mathbb{R}^n\).

Let \(\| \cdot \|_0\) denote the Euclidean norm.

\[S^n = \{ x \in \mathbb{R}^n : \| x \|_0 = 1 \} \text{ is compact}.\]

Thus \(\exists x_0 \in S\) such that \(m = \| x_0 \|_0 \leq \| x \|\).

Thus \(\| x_0 \|_0 \leq 1\).

Remark that \(m \neq 0\) since \(m = 0 \implies \| x_0 \| = 0\) and \(x_0 = 0\) is impossible as \(\| x_0 \|_0 = 1\).
For $x \in \mathbb{R}^n$, $x \neq 0$,

$$\frac{x}{\|x\|} \in S^1, \quad m \leq \varphi\left(\frac{x}{\|x\|}\right).$$

$$m \|x\| \leq \frac{x \cdot v_i}{\|x\|},$$

$$\|x\| m \leq \sum_{i} x_i v_i \|v_i\| = \|x\| \|v_i\|.$$ 

Now for $x = \sum \frac{x_i}{v_i}, v_i \neq 0$

$$\|f(\sum x_i v_i)\| = \|\sum \frac{x_i}{v_i} f(v_i)\|$$

$$\leq \sum \left|\frac{x_i}{v_i}\right| \|f(v_i)\|$$

$$\leq \sum \frac{|x_i|}{v_i} \|f(v_i)\|$$

$$\leq \frac{\|x\|}{m} \|v_i\| \|f(v_i)\|$$

Thus

$$\frac{\|f(x)\|}{\|x\|} \leq \sum \frac{1}{m} f(v_i)$$

Thus $f$ is continuous by part b.}
Checking that the norm makes the set of infinite sequences above a normed vector space:

1. Given that at least one of the \( x_i \)'s is nonzero, clearly \( \max\{|x_1|, |x_2|, |x_3|, \ldots\} > 0 \).

2. \( \max\{|x_1|, |x_2|, |x_3|, \ldots\} = 0 \) only if all the \( x_i \)'s are zero.

3. \( \|cx_1, cx_2, cx_3, \ldots\| = \max\{|cx_1|, |cx_2|, |cx_3|, \ldots\} = \max\{|c||x_1|, |c||x_2|, |c||x_3|, \ldots\} = |c| \cdot \max\{|x_1|, |x_2|, |x_3|, \ldots\} \).

4. By the triangle inequality, \( |x_i + y_i| \leq |x_i| + |y_i| \leq \max_i(x_i) + \max_i(y_i) \). So we must have \( \max_i(x_i + y_i) \leq \max_i(x_i) + \max_i(y_i) \).

Showing the map is a one-to-one linear transformation:

Within each component, we have \( f(x_i) = i \cdot x_i \). This is clearly a linear transformation, and one-to-one. Since all components are then mapped by a linear transformation, the larger map is a linear transformation.

Assume \( (x_1, 2x_2, 3x_3, \ldots) = (x_1', 2x_2', 3x_3', \ldots) \)

Then \( i \cdot x_i = i \cdot x_i \) for \( i = 1, 2, \ldots \)

so \( x_i = x_i' \)

and \( (x_1, x_2, x_3, \ldots) = (x_1', x_2', \ldots) \)

so the map is one-to-one.
Set \( X^n = (x^1, x^2, \ldots) \)
when \( x^i = 0 \) if \( i \neq n \)
\( x^n = 1 \) if \( i = n \)

\[ x^1 = (1, 0, 0, 0, \ldots) \]
\[ x^2 = (0, 1, 0, 0, \ldots) \]
\[ x^3 = (0, 0, 1, 0, \ldots) \]

If \( f \) denotes the map: \( (x^1, x^2, \ldots) \rightarrow (x^1, 2x^2, 3x^3, \ldots) \)

\[ f(x^n) = (x^n, 2x^n, \ldots, \) \( ) = (0, \ldots, 0, n, 0, \ldots) \]

Thus \( \| f(x^n) \| = n \) and \( \| x^n \| = 1 \).

So \( \| f(x^n) \| = n \)
\[ \frac{\| f(x^n) \|}{\| x^n \|} \]

and the set \( \{ \| f(x) \| / \| x \| \}, x \rightarrow 0 \) \( f \) is not bounded

and then \( f \) is not
23. Use Problem 22 to prove that if \( V \) is a finite dimensional vector space over \( \mathbb{R} \) and \( \| \|_1, \| \|_2 \) are two norm functions on \( V \).

**Solution**

Let \( e_i \) be the canonical base of \( V \), and let \( x \in V \). Then \( x = \sum_{i=1}^{n} x_i e_i = \sum_{i=1}^{n} x_i e_i \leq \sum_{i=1}^{n} \| x_i \|_1, \| e_i \|_1 \) by using the definition of a norm and \( \| \|_1 \) is any norm. We can apply the Cauchy-Schwarz inequality \( \| x \|_2 \leq \sum_{i=1}^{n} \| x_i \|_1 \| e_i \|_1 \leq \sum_{i=1}^{n} \| x_i \|_2 \| e_i \|_2 \). We can see that \( \sum_{i=1}^{n} \| x_i \|_2^2 = \| x \|_2^2 \) and \( \sum_{i=1}^{n} \| e_i \|_2^2 = \mu_2 \) where \( \| \|_2 \) is the euclidean norm, and \( \mu_2 \) is some constant. Then \( \| x \|_1 \leq \mu_1 \| x \|_2 \). Now we show that \( f(x) = \| x \|_1 \) is continuous with respect to the euclidean norm. Let \( \epsilon > 0 \), we need to show

\[
\exists \delta > 0 \text{ s.t. } \| x - y \|_2 < \delta \Rightarrow \| x \|_1 - \| y \|_1 < \epsilon. \quad \text{Let } \delta = \frac{\epsilon}{\mu_1}. \text{ Then }
\]

\[
\frac{\epsilon}{\mu_1} \geq \| x - y \|_2 \geq \| x \|_2 - \| y \|_2 = \frac{\epsilon}{\mu_2} \geq \| x \|_1 - \| y \|_1 = \frac{\epsilon}{\mu_1} \| x \|_2 - \| y \|_1. \quad \text{Then, } \epsilon > \| x \|_1 - \| y \|_1, \quad \text{F is continuous. Let } S = \{ x \in V \text{ s.t. } \| x \|_2 = 1 \}. \text{ S is closed and bounded, so it is compact, and then F has a minimum value in S at } x_{\text{min}}. \text{ Now } \forall z \in S, \| x_{\text{min}} - z \|_2 \leq \| x_{\text{min}} - z \|_1 \geq m_1, \text{ where } m_1 \text{ is the minimum value of F in } z. \text{ Then, } \| z \|_1 \geq m_1 \| z \|_2. \quad \text{For any norms, we have basically shown that } \exists \mu_1, m_2, \mu_2 \text{ such that } m_1 \| x \|_2 \leq \| x \|_1 \leq \| x \|_e = m_1 \| x \|_2 \leq \mu_2 \| x \|_1 \leq m_2 \mu_2. \quad \text{It basically follows that given that } \mathbb{R} \text{ is complete, and all norms are equivalent to the euclidean norm, the space is complete V norm.}
\]

33)

a. Show that the sequence of functions \( x, x^2, x^3, \ldots \) converges uniformly on \([0, a]\) for any \( a \in (0, 1) \), but not on \([0, 1]\).

Let \( \{ f_n \} = \{ x^n \} \), and suppose \( f^n \to f \). We must show that for \( \epsilon > 0 \), \( \exists N \) such \( d(f, f^n) < \epsilon \) whenever \( n > N \) for all \( x \).

For \( a \in (0, 1) \), it is clear to see that \( x^n \to 0 \) as \( n \) approaches infinity. We must then show \( \| x^n \| < \epsilon \) whenever \( n \) is greater than some \( N \).

On \([0, a]\), \( x^n \) attain its max at \( x = a \), so \( x^n < a^n \). Then note \( a^n \) decreases with increasing \( n \), so we choose \( N \) such \( a^n < \epsilon \).

\( \{ f_n \} \) doesn't converge uniformly on \([0, 1]\) because at \( x = 1 \) \( f^n = 1^n = 1 \neq a \) for all \( n \).

b. Show that the sequence of functions \( x(1 - x), x^2(1 - x), x^3(1 - x), \ldots \) converges uniformly on \([0, 1]\).

Since on \([0, 1]\), at least one of the quantities \( x^n \) and \( (1 - x) \) is less than 1, and at is at most 1, thus we might guess \( f = 0 \). Then we must show for \( \epsilon > 0 \), \( \exists N \) such \( |x^n(1 - x)| < \epsilon \) if \( n > N \).

\( x^n \) and \( (1 - x) \) are both continuous functions, and using calculus, we can calculate the maximum value that \( |x^n(1 - x)| \) attains on \([0, 1]\).

\[
\frac{d}{dx} (x^n(1 - x)) = nx^n - x^{n+1} = -x^n + nx^{n-1} - nx^n = -x + n - nx = 0
\]

So \( x^n(1 - x) \) attains its max at \( x = \frac{n}{n+1} \), which is \( \left( \frac{n}{n+1} \right)^n \left( \frac{1}{n+1} \right) \).
Then $|x^n(1-x)| < \left(\frac{n}{n+1}\right)^n \left(\frac{1}{n+1}\right) < \frac{1}{n+1} < \varepsilon$. If we choose $N = (1-\varepsilon)/\varepsilon$, then whenever $n > N$, we will have $|x^n(1-x)| < \varepsilon$.

34) Is the sequence of functions $f_1, f_2, f_3, \ldots$ on $[0,1]$ uniformly convergent if $f_n(x) = \frac{x}{1+nx^2}$. Note $f_n \to 0$. We must find for $\varepsilon > 0$, an $N$ such $n > N$ implies $|f_n - 0| < \varepsilon$.

Observe $f'_n = \frac{(1+nx^2)-x(2nx)}{(1+nx^2)^2} = \frac{1-nx^2}{(1+nx^2)^2} = 0$. This has solution at $x = \sqrt{\frac{1}{n}}$, and $f_n$ attains a max value $\frac{1}{2}\sqrt{\frac{1}{n}}$. So $\left|\frac{x}{1+nx^2}\right| < \frac{1}{2}\sqrt{\frac{1}{n}}$, thus if we choose $N > 1/4\varepsilon^2$, we will have $|f_n - 0| < \varepsilon$ whenever $n > N$.

$f_n(x) = \frac{nx}{1+nx^2}$. Note for all $n$, $f_n(0) = 0$, and for $x > 0$, $f_n \to \frac{1}{x}$. Since $\lim$ $f_n$ is not continuous on $[0,1]$, it does not converge uniformly.

$f_n(x) = \frac{nx}{1+nx^2}$. Again we have $f_n \to 0$. We must find for $\varepsilon > 0$, an $N$ such $n > N$ implies $|f_n - 0| < \varepsilon$.

Again, taking the derivative and setting it to 0 gives us:

$$f'_n = \frac{(1+n^2x^2)n-nx(2n^2x)}{(1+n^2x^2)^2} = \frac{n-n^3x^2}{(1+n^2x^2)^2} = 0$$

This has solution at $x = 1/n$. But this means that $f_n$ attains a max of $f_n\left(\frac{1}{n}\right) = \frac{n(\frac{1}{n})}{1+n^2(\frac{1}{n})} = \frac{1}{2}$. Thus it would not be possible to choose an $N$ for all $\varepsilon$, specifically any $\varepsilon < \frac{1}{2}$.

37) Let $f_1, f_2, f_3, \ldots$ and $g_1, g_2, g_3, \ldots$ be uniformly convergent sequences of real-valued functions on a metric space $E$. Show that the sequence $f_1 + g_1, f_2 + g_2, \ldots$ is uniformly convergent.

Let $f_n \to f$ and $g_n \to g$. We have that for all $x$, for $\varepsilon > 0$, $\exists N_1, N_2$ such $|f_n(x) - f(x)| < \frac{\varepsilon}{2}$ whenever $n > N_1$ and $|g_n(x) - g(x)| < \frac{\varepsilon}{2}$ whenever $n > N_2$.

We hypothesize that $f_n + g_n \to f + g$. So we must find $N$ such $|(f_n + g_n)(x) - (f + g)(x)| < \varepsilon$ whenever $n > N$. Note $|(f_n + g_n)(x) - (f + g)(x)| = |(f_n - f)(x) + (g_n - g)(x)| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)|$. So if we take $N = \max(N_1, N_2)$, we will have that $|(f_n + $
Given \( \epsilon > 0 \), there exists some \( N \) such that \( d(f_n(x), f(x)) < \epsilon \) when \( n > N \). Since each function is bounded, all elements of \( f_n(x) \) are bounded and thus all elements in an open ball \( B_r(x_0) \). Which means that we have

\[
 d(f(x), y) \leq d(f(x), f_n(x)) + d(f_n(x), x_0) < \epsilon + r, \quad \text{for some ball } B_r(x_0).
\]

Thus all elements are contained in an open ball \( B_{\epsilon+r}(x_0) \). Therefore the sequence is bounded.