1.1. Prove the following facts, which were left as exercises.

(a) For a sequence of sets \( \{E_k\} \), \( \limsup E_k \) consists of those points which belong to infinitely many \( E_k \), and \( \liminf E_k \) consists of those points which belong to all \( E_k \) from some \( k \) on.

\[ \text{Solution. Suppose } x \in \limsup E_k. \text{ Then } x \in \bigcup_{k \geq j} E_k \text{ for all } j, \text{ so for every } j \geq 1, \text{ there exists } k \geq j \text{ such that } x \in E_i; \text{ hence } x \in E_k \text{ for infinitely many } k. \text{ All of these implications are reversible.} \]

Suppose \( x \in \liminf E_k \). Then \( x \in \bigcap_{k \geq j} E_k \) for some \( k \). Thus, for some \( j \geq 1, x \in E_k \) for all \( k \geq j \); hence \( x \in E_k \) for all but finitely many \( k \). All of these implications are reversible. \( \square \)

(b) The De Morgan laws:

\[ \left( \bigcup_{E \in \mathcal{F}} E \right)^c = \bigcap_{E \in \mathcal{F}} E^c \text{ and } \left( \bigcap_{E \in \mathcal{F}} E \right)^c = \bigcup_{E \in \mathcal{F}} E^c. \]

\[ \text{Solution. Suppose } x \in (\bigcup_{E \in \mathcal{F}} E)^c; \text{ then } x \notin \bigcup_{E \in \mathcal{F}} E, \text{ so } x \text{ fails to be in every } E \in \mathcal{F}; \text{ hence } x \in E^c \text{ for every } E \in \mathcal{F}, \text{ so } x \in \bigcap_{E \in \mathcal{F}} E^c. \text{ All of these implications are reversible.} \]

Suppose \( x \in (\bigcap_{E \in \mathcal{F}} E)^c \); then \( x \notin \bigcap_{E \in \mathcal{F}} E, \text{ so } x \text{ fails to be some } E \in \mathcal{F}; \text{ hence } x \in E^c \) for some \( E \in \mathcal{F}, \text{ so } x \in \bigcup_{E \in \mathcal{F}} E^c. \text{ All of these implications are reversible.} \]

[Note: \( \mathcal{F} \) is an arbitrary family of sets and may have infinite size, so a proof by induction cannot work here.] \( \square \)

(d) Theorem 1.4:

(a) \( L = \limsup_{k \to \infty} a_k \) if and only if (i) there is a subsequence \( \{a_{k_j}\} \) of \( \{a_k\} \) which converges to \( L \) and (ii) if \( L' > L \), there is an integer \( K \) such that \( a_k < L' \) for \( k \geq K \).

(b) \( \ell = \liminf_{k \to \infty} a_k \) if and only if (i) there is a subsequence \( \{a_{k_j}\} \) of \( \{a_k\} \) which converges to \( \ell \) and (ii) if \( \ell' < \ell \), there is an integer \( K \) such that \( a_k > \ell' \) for \( k \geq K \).

\[ \text{Solution.} \]

(a) First, the forward direction. Let \( L_j = \sup_{k \geq n} a_k \), so \( L = \lim_{n \to \infty} L_n \). Given \( j \geq 1 \), choose \( n_j \geq j \) such that \( |L_{n_j} - L| < \frac{1}{2j} \) and choose \( k_j \geq n_j \) such that \( |a_{k_j} - L_{n_j}| < \frac{1}{2j} \). Then

\[ |a_{k_j} - L| \leq |a_{k_j} - L_{n_j}| + |L_{n_j} - L| < \frac{1}{j}, \]

so \( a_{k_j} \to L \). Now, given \( L' > L \), choose \( K \) such that \( |L_K - L| < L' - L \); then \( \sup_{k \geq K} a_k < L' \), so \( a_k < L' \) for all \( k \geq K \).

Now, the reverse direction. Let \( \varepsilon > 0 \); then there exists \( J \) such that for every \( j \geq J \), \( |a_{k_j} - L| < \varepsilon \). Thus, \( \sup_{k \geq j} a_k = L_j \geq L - \varepsilon \); thus, \( \lim_{j \to \infty} L_j \geq L \).

On the other hand, there exists \( K \) such that for every \( k \geq K \), \( a_k < L + \varepsilon \); thus, \( \sup_{k \geq K} a_k = L_K \leq L + \varepsilon \), so \( \lim_{j \to \infty} L_j \leq L \). Thus, \( \limsup a_k = \lim_{j \to \infty} L_j = L \).
(b) Recall that \( \liminf a_k = - \limsup(-a_k) \). Using the result of part (a), if \( \ell = \liminf a_k \), then \(-\ell = \limsup(-a_k)\), so there is a subsequence \( \{-a_{k_j}\} \) of \(-a_k\) converging to \(-\ell\) (thus a subsequence \( \{a_{k_j}\} \) of \(a_k\) converging to \(\ell\)), and if \(\ell' < \ell\), then \(-\ell' > \ell\), so there exists \(K\) such that \(-a_k < -\ell'\) (hence \(a_k > \ell'\)).

On the other hand, given \(\{a_k\}\), if there is a subsequence \(\{a_{k_j}\}\) converging to \(\ell\) and for all \(\ell' < \ell\) there exists \(K\) such that \(a_{k_j} > \ell'\) whenever \(k \geq K\), then the hypothesis of the reverse implication of part (a) is satisfied: there is a subsequence \(\{-a_{k_j}\}\) of \(-a_k\) which converges to \(-\ell\) and, if \(-\ell' > \ell\), there is an integer \(K\) such that \(-a_k < -\ell'\) for \(k \geq K\). Thus \(-\ell = \limsup(-a_k)\), so \(\ell = \liminf a_k\).

\[ \square \]

(n) Theorem 1.14:

(a) \(M = \limsup_{x \to x_0, x \in E} f(x)\) if and only if (i) there exists \(\{x_k\}\) in \(E\) such that \(x_k \to x_0\) and \(f(x_k) \to M\), and (ii) if \(M' > M\), there exists \(\delta > 0\) such that \(f(x) < M'\) for \(x \in B'(x_0; \delta) \cap E\).

(b) \(m = \liminf_{x \to x_0, x \in E} f(x)\) if and only if (i) there exists \(\{x_k\}\) in \(E\) such that \(x_k \to x_0\) and \(f(x_k) \to m\), and (ii) if \(m' < m\), there exists \(\delta > 0\) such that \(f(x) > m'\) for \(x \in B'(x_0; \delta) \cap E\).

**Solution.** For simplicity, let \(B_\delta\) denote the open ball of radius \(\delta\) centered at \(x_0\) intersected with \(E\).

(a) First, the forward direction. Let \(M_\delta = \sup_{B_\delta} f(x)\), so \(M = \lim_{\delta \to 0} M_\delta\). Given \(k \geq 1\), choose \(\delta_k \leq \frac{1}{j}\) such that \(|M_{\delta_k} - M| < \frac{1}{2^k}\), and choose \(x_k \in B_{\delta_k}\) such that \(|f(x_k) - M_{\delta_k}| < \frac{1}{2^k}\). Then

\[ |f(x_k) - M| \leq |f(x_k) - M_{\delta_k}| + |M_{\delta_k} - M| < \frac{1}{k}, \]

so \(f(x_k) \to M\) (and \(x_k \to x_0\) since \(|x_k - x_0| \leq \frac{1}{k}\)). Now, given \(M' > M\), choose \(\delta\) such that \(|M_\delta - M| < M' - M\); then \(\sup_{B_\delta} f(x) < M'\), so \(f(x) < M'\) for all \(x \in B_\delta\).

Now, the reverse direction. Let \(\varepsilon > 0\); then there exists \(K\) such that for every \(k \geq K\), \(|f(x_k) - M| < \varepsilon\). Thus, letting \(\delta_k = |x_k - x_0|\), we have \(\sup_{B_{\delta_k}} f(x) = M_{\delta_k} \geq M - \varepsilon\); thus, \(\lim_{\delta \to 0} M_\delta \geq M\). On the other hand, there exists \(\delta\) such that for every \(x \in B_\delta\), \(f(x) < M + \varepsilon\); thus, \(\sup_{B_\delta} f(x) = M_\delta \leq M + \varepsilon\), so \(\lim_{\delta \to 0} M_\delta \leq M\). Thus, \(\lim \sup f(x) = \lim_{\delta \to 0} M_\delta = M\).

(b) Recall that \(\liminf f(x) = -\limsup(-f(x))\). Using the result of part (a), if \(m = \liminf f(x)\), then \(-m = \limsup(-f(x))\), so there is a sequence \(\{-f(x_k)\}\) (with \(x_k \to x_0\)) converging to \(-m\) (thus a sequence \(\{f(x_k)\}\) with \(x_k \to x_0\) and \(f(x_k) \to m\)), and if \(m' < m\), then \(-m' > m\), so there exists \(\delta\) such that \(-f(x) < -m'\) (hence \(f(x) > m'\)) for all \(x \in B_\delta\).
On the other hand, if there is a sequence \( \{f(x_k)\} \) with \( x_k \to x_0 \) and \( f(x_k) \to m \) and for all \( m' < m \) there exists \( \delta \) such that \( f(x) > m' \) whenever \( x \in B_\delta \), then the hypothesis of the reverse implication of part (a) is satisfied: there is a sequence with \( x_k \to x_0 \) and \( -f(x_k) \to -m \) and, if \( m' > -m \), there exists \( \delta \) such that \( -f(x) < -m' \) for \( x \in B_\delta \). Thus \( -m = \lim sup(-f(x)) \), so \( m = \lim inf f(x) \).

\[ \square \]

1.2. Find \( \lim sup E_k \) and \( \lim inf E_k \) if \( E_k = [-1/k, 1] \) for \( k \) odd and \( E_k = [-1, 1/k] \) for \( k \) even.

\textit{Solution.} Recall that \( \lim sup E_k \) is the set of points which appear in infinitely many \( E_k \), and \( \lim inf E_k \) is the set of points which appear in all but finitely many \( E_k \). If \( x \in [-1, 0] \), then \( x \in E_k \) for all even \( k \), and if \( x \in [0, 1] \), then \( x \in E_k \) for all odd \( k \), so \( \lim sup E_k = [-1, 1] \). If \( x \in [-1, 0) \), then \( x \notin E_k \) for odd \( k \) when \( x < -1/k \), which must eventually happen for large enough \( k \); similarly, if \( x \in (0, 1] \), then \( x \notin E_k \) for even \( k \) when \( x > 1/k \). Thus, every point but \( x = 0 \) fails to appear in infinitely many \( E_k \), so \( \lim inf E_k = \{0\} \).

\[ \square \]

1.3.

(a) Show that \( (\lim sup E_k)^c = lim inf E_k^c \).

(b) Show that if \( E_k \not\supset E \) or \( E_k \not\subset E \), then \( \lim sup E_k = \lim inf E_k = E \).

\textit{Solution.} (a) This follows from the De Morgan laws:

\[
(\lim sup E_k)^c = \left( \bigcap_{j=1}^{\infty} \left( \bigcup_{k \geq j} E_k \right) \right)^c = \bigcup_{j=1}^{\infty} \left( \bigcap_{k \geq j} E_k^c \right) = \lim inf E_k^c.
\]

(b) \( E_k \not\supset E \) means that \( E_k \subseteq E_{k+1} \) and \( \bigcup E_k = E \); in this case,

\[
\lim inf E_k = \bigcup_{j=1}^{\infty} \left( \bigcap_{k \geq j} E_k \right) = \bigcup_{j=1}^{\infty} E_j = E \quad \text{and} \quad \lim sup E_k = \bigcap_{j=1}^{\infty} \left( \bigcup_{k \geq j} E_k \right) = \bigcap_{j=1}^{\infty} E = E.
\]

On the other hand, \( E_k \not\subset E \) means that \( E_k \supseteq E_{k+1} \) and \( \bigcap E_k = E \); in this case,

\[
\lim inf E_k = \bigcup_{j=1}^{\infty} \left( \bigcap_{k \geq j} E_k \right) = \bigcup_{j=1}^{\infty} E = E \quad \text{and} \quad \lim sup E_k = \bigcap_{j=1}^{\infty} \left( \bigcup_{k \geq j} E_k \right) = \bigcap_{j=1}^{\infty} E_j = E.
\]

\[ \square \]

1.9. Prove that any closed subset of a compact set is compact.

\textit{Solution.} Let \( X \) be a compact space, let \( F \subseteq X \) be closed in \( X \), and let \( \mathcal{C} = \{G_\alpha\} \) be an open cover of \( F \); then each \( G_\alpha = G_\alpha^c \cap F \) for a subset \( G_\alpha^c \subseteq X \) which is open in \( X \). Since \( X = F \cap F^c \) and \( \bigcup G_\alpha^c \supseteq F \), \( F^c \cup \bigcup G_\alpha^c = X \): moreover, since \( F \) is closed in \( X \), \( F^c \) is open in \( X \). Thus, \( \mathcal{C}' = \{G_\alpha^c\} \cup \{F^c\} \) is an open cover of \( X \).
Since \( X \) is compact, there exists a finite subcover \( C'' = \{G'_1, ..., G'_n\} \subseteq C' \) such that 
\[
\bigcup_{k=1}^n G'_k = X; \text{ hence } \bigcup_{k=1}^n G'_k \supseteq F. \]
Now, each \( G'_k \) has the property that either \( G'_k \cap F = G'_k \in C \) or \( G'_k \cap F = \emptyset \) (if \( G'_k = F^c \)). Thus, we can take \( C''' = \{G_1, G_2, ..., G_n\} \setminus \{\emptyset\} \); then 
\[
\bigcup_{G \in C''} G = F \text{ and } C'' \subseteq C, \text{ so } C''' \text{ is a finite open subcover of } C. \]
Hence \( F \) is a compact subspace of \( X \).

[Note that \( X \) need not have a norm, so arguments involving boundedness would fail in such a case.] □

1.16. If \( \{f_k\} \) is a sequence of bounded, Riemann-integrable functions on an interval \( I \) which converges uniformly on \( I \) to \( f \), show that \( f \) is Riemann integrable on \( I \) and that
\[
(R) \int_I f_k(x) \, dx \to (R) \int_I f(x) \, dx.
\]

Solution. Given \( \varepsilon > 0 \), choose \( k \) such that for every \( x \in I \), we have
\[
|f_k(x) - f(x)| < \varepsilon;
\]
it follows that
\[
\left| \sup_{I} f_k(x) - \sup_{I} f(x) \right| < \varepsilon \quad \text{and} \quad \left| \inf_{I} f_k(x) - \inf_{I} f(x) \right| < \varepsilon.
\]
Since \( f_k \) is Riemann-integrable, there exists a partition \( \Gamma \) such that
\[
U_{\Gamma}(f_k) - L_{\Gamma}(f_k) < \varepsilon.
\]
Then
\[
|U_{\Gamma}(f) - L_{\Gamma}(f)| \leq |U_{\Gamma}(f) - U_{\Gamma}(f_k)| + |U_{\Gamma}(f_k) - L_{\Gamma}(f_k)| + |L_{\Gamma}(f_k) - L_{\Gamma}(f)| \leq 3\varepsilon |I|.
\]
Hence \( f \) is also Riemann-integrable. Moreover,
\[
\left| (R) \int_I f_k(x) \, dx - (R) \int_I f(x) \, dx \right| \leq (R) \int_I |f_k(x) - f(x)| \, dx < \varepsilon |I|.
\]
Hence the integral of \( f \) is the limit of the integrals of the \( f_k \). □