9.1. Use Minkowski’s integral inequality to prove (9.1): if $1 \leq p \leq \infty$, $f \in L^p(\mathbb{R}^n)$, and $g \in L^1(\mathbb{R}^n)$, then $f \ast g \in L^p(\mathbb{R}^n)$ and $\|f \ast g\|_p \leq \|f\|_p \|g\|_1$.

Solution. The cases of $p = 1, \infty$ can be dealt with as in the text. For $1 < p < \infty$, Minkowski’s integral inequality states that

$$\left\| \int f(x, y) \, dx \right\|_p \leq \int \|f(x, y)\|_p \, dx,$$

where the norms are taken with respect to the $y$-variable. Letting $F(x, y) = f(y - x) g(x)$, we have

$$f \ast g(y) = \int F(x, y) \, dx,$$

so

$$\|f \ast g\|_p = \left\| \int F(x, y) \, dx \right\|_p \leq \int \|f(y - x) g(x)\|_p \, dx = \int |g(x)| \|f(y - x)\|_p \, dx = \|f\|_p \|g\|_1.$$ 

The penultimate equality follows because $g$ is a function of $x$, but the norm is taken with respect to $y$. The last equality follows because integrals over $\mathbb{R}^n$ are invariant under translations. \qed
9.3. Show that if \( f \in L^p(\mathbb{R}^n) \) and \( K \in L^q(\mathbb{R}^n) \), \( 1 \leq p \leq \infty \) and \( 1/p + 1/q = 1 \), then \( f \ast K \) is bounded and continuous in \( \mathbb{R}^n \).

Solution. By Young’s convolution theorem (see Problem 9.2), \( ||f \ast K||_{\infty} \leq ||f||_p ||K||_q < +\infty \), so \( f \ast K \) is bounded by \( c = ||f \ast K||_{\infty} \) a.e. in \( \mathbb{R}^n \). (Showing that \( f \ast K \) is continuous will prove that \( f \ast K \) is bounded everywhere by \( c \).)

If \( 1 < p \leq +\infty \), then by Hölder’s inequality

\[
|f \ast K(x+h) - f \ast K(x)| \leq \int |f(t)| |K(x+h-t) - K(x-t)| \, dt \leq ||f||_p \left\| \tilde{K}(t-h) - \tilde{K}(t) \right\|_q,
\]

where \( \tilde{K}(t) = K(x-t) \). Since \( \tilde{K} \in L^q \), we have by continuity in \( L^q \) that \( \left\| \tilde{K}(t-h) - \tilde{K}(t) \right\|_q \to 0 \) as \( |h| \to 0 \). Since \( ||f||_p < +\infty \), we’ve proven continuity of \( f \ast K \).

If \( p = 1 \) (so that \( q = +\infty \)), then switch the roles of \( K \) and \( f \):

\[
|f \ast K(x+h) - f \ast K(x)| \leq \int |K(t)||f(x+h-t) - f(x-t)| \, dt \leq ||K||_{\infty} \left\| \tilde{f}(t-h) - \tilde{f}(t) \right\|_1 \to 0.
\]
9.5. Let $G, G_1$ be bounded open subsets of $\mathbb{R}^n$ such that $\overline{G_1} \subset G$. Construct a function $h \in C_0^\infty$ such that $h = 1$ in $G_1$ and $h = 0$ outside $G$. [Hint: Choose an open $G_2$ such that $\overline{G_1} \subset G_2$ and $\overline{G_2} \subset G$. Let $h = 1_{G_2} \ast K$ for a $K \in C^\infty$ with suitably small support and $\int K = 1$.

Solution. Let
\[ K(x) = Ce^{-1/(1-|x|^2)}\mathbf{1}_{B_1(0)}, \]
where $C$ is chosen so that $\int K = 1$. Choose $\varepsilon < \min(\text{dist}(G_1, \partial G_2), \text{dist}(G_2, \partial G))$, and define $K_\varepsilon = \frac{1}{\varepsilon^n} K(x/\varepsilon)$, so that $\int K_\varepsilon = 1$. Also, supp $K_\varepsilon = B_\varepsilon(0)$ and $K_\varepsilon \in C_0^\infty$.

Choose $G_2$ as described in the hint, and define $h(x) = 1_{G_2} \ast K_\varepsilon(x)$. Since $1_{G_2} \in L^1$ (as $G_2$ is bounded) and $K \in C_0^\infty$, we have $h \in C_0^\infty$. Also, for $x \in G_1$, we have
\[ h(x) = \int_{\mathbb{R}^n} 1_{G_2}(x-t)K(t) \, dt = \int_{B_\varepsilon(0)} 1_{G_2}(x-t)K(t) \, dt = \int_{B_\varepsilon(0)} K(t) \, dt = 1 \]
since $1_{G_2}(x-t) = 1$ for such points and $|t| < \varepsilon$; if $x \notin G$, then we have
\[ h(x) = \int_{\mathbb{R}^n} 1_{G_2}(x-t)K(t) \, dt = \int_{B_\varepsilon(0)} 1_{G_2}(x-t)K(t) \, dt = 0 \]
since $1_{G_2}(x-t) = 0$ for such points and $|t| < \varepsilon$. \qed
The maximal function is defined as $f^*(x) = \sup |Q|^{-1} \int_Q |f|$, where the supremum is taken over cubes $Q$ with center $x$. Let $f^{**}(x)$ be defined similarly, but with the supremum taken over all $Q$ containing $x$. Thus, $f^*(x) \leq f^{**}(x)$. Show that there is a positive constant $c$ depending only on the dimension such that $f^{**}(x) \leq cf^*(x)$.

**Solution.** Write $Q = [a_1, b_1] \times \cdots \times [a_n, b_n]$. For each $i = 1, \ldots, n$ let $\delta_i = \max(x_i - a_i, b_i - x_i)$, and let $\delta = \max_i \delta_i$. Let $Q'$ be the cube centered at $x$ with each side length $2\delta$: then $Q \subset Q'$, so

$$\int_Q |f| \leq \int_{Q'} |f|,$$

and $|Q'| \leq 2^n |Q|$, so

$$\frac{1}{|Q|} \int_Q |f| \leq \frac{2^n}{|Q'|} \int_{Q'} |f|.$$

So, for every $Q$ containing $x$, there exists $Q'$ centered at $x$ such that the above holds. Thus, $f^{**}(x) \leq 2^n f^*(x)$. □