Homework 3

1. If the inverse $T^{-1}$ of a closed linear operator exists, show that $T^{-1}$ is a closed linear operator.

Solution: Assuming that the inverse of $T$ were defined, then we will have to have that $D(T^{-1}) = \text{Ran}(T)$. Suppose that $\{u_n\} \in D(T^{-1})$ is a sequence such that $u_n \to u$ and $T^{-1}u_n \to x$. We need so that that $u \in D(T^{-1})$ and $T^{-1}u = x$. Then by the simple criteria, we would have that $T^{-1}$ is closed.

Since $u_n \in D(T^{-1})$ we have that $u_n = Tx_n$, where $x_n \in D(T)$. Also, note we have that $T^{-1}u_n = x_n \to x$. So, we have a sequence of vectors $\{x_n\} \in D(T)$ with $x_n \to x$ and $Tx_n \to u$. Since $T$ is closed, we have that $x \in D(T)$ and $Tx = u$. But, this last equality implies that $u \in D(T^{-1})$ and that $x = T^{-1}u$, which is what we wanted.

2. Let $T$ be a closed linear operator. If two sequences $\{x_n\}$ and $\{\tilde{x}_n\}$ in the domain of $T$ both converge to the same limit $x$, and if $\{Tx_n\}$ and $\{T\tilde{x}_n\}$ both converge, show that $\{Tx_n\}$ and $\{T\tilde{x}_n\}$ both have the same limit.

Solution: Suppose that $\{x_n\}, \{\tilde{x}_n\} \in D(T)$ are such that $x_n \to x$ and $\tilde{x}_n \to x$. Let $Tx_n \to y$ and $T\tilde{x}_n \to \tilde{y}$. We need to show that $y = \tilde{y}$.

Since $T$ is closed, then we know that for $x_n \to x$ and $Tx_n \to y$ implies that $x \in D(T)$ and $Tx = y$. Similarly, we have that $\tilde{x}_n \to x$ and $T\tilde{x}_n \to \tilde{y}$ implies that $x \in D(T)$ and $Tx = \tilde{y}$. But, this then gives $y = Tx = \tilde{y}$, and we they have the same limit as claimed.

3. Let $X$ and $Y$ be Banach spaces and $T : X \to Y$ an injective bounded linear operator. Show that $T^{-1} : \text{Ran}T \to X$ is bounded if and only if $\text{Ran}T$ is closed in $Y$.

Solution: First suppose that $\text{Ran}T$ is closed in $Y$. By problem 1 we know that $T^{-1}$ is closed and $D(T^{-1}) = \text{Ran}(T)$. Then by the Closed Graph Theorem, we have that $T^{-1}$ is a bounded linear operator from $\text{Ran}T \to X$.

Now, suppose that $T^{-1} : \text{Ran}T \to X$ is bounded, i.e., $\|T^{-1}y\|_X \leq c\|y\|_Y$. The goal is to show that $y \in \text{Ran}T$. Let $y \in \overline{\text{Ran}T}$, and then there exists $x_n \in X$ such that $Tx_n := u_n \to y$. We then have that the sequence $\{x_n\}$ is Cauchy since

$$\|x_n - x_m\|_X = \|T^{-1}Tx_n - T^{-1}Tx_m\|_X = \|T^{-1}(u_n - u_m)\|_X \leq c\|u_n - u_m\|_Y$$

and since $u_n \to y$ this gives that the claim. Since $X$ is complete, let $x$ denote the limit of the sequence $\{x_n\}$. Then we will show that $y = Tx \in \text{Ran}T$, and so the range of $T$ is
closed. But, note that
\[
\|y - Tx\|_Y = \|y - u_n + u_n - Tx\|_Y \\
\leq \|y - u_n\|_Y + \|u_n - Tx\|_Y \\
\leq \|y - u_n\|_Y + \|T\| \|x_n - x\|_X.
\]
Here we have used that \(T\) is bounded. But, the expression on the right hand side can be made arbitrarily small, and so \(y = Tx\) as desired.

4. Let \(T : X \to Y\) be a bounded linear operator, where \(X\) and \(Y\) are Banach spaces. If \(T\) is bijective, show that there are positive real numbers \(a\) and \(b\) such that
\[
a \|x\|_X \leq \|Tx\|_Y \leq b \|x\|_X \quad \forall x \in X.
\]

**Solution:** Since \(T\) is a bounded linear operator, then we immediately have that
\[
\|Tx\|_Y \leq \|T\| \|x\|_X \quad \forall x \in X.
\]
However, since \(T\) is bijective and bounded we have that \(T^{-1}\) is bounded as well. Note that \(T^{-1} : Y \to X\), so for any \(y \in Y\) we have that
\[
\|T^{-1}y\|_X \leq \|T^{-1}\| \|y\|_Y.
\]
Apply this when \(y = Tx\), and we then have
\[
\|x\|_X \leq \|T^{-1}\| \|Tx\|_Y.
\]
Combining inequalities we have that
\[
a \|x\|_X \leq \|Tx\|_Y \leq b \|x\|_X \quad \forall x \in X
\]
with \(a = \|T^{-1}\|^{-1}\) and \(b = \|T\|\).

5. Let \(X_1 = (X, \|\cdot\|_1)\) and \(X_2 = (X, \|\cdot\|_2)\) be Banach spaces. If there is a constant \(c\) such that \(\|x\|_1 \leq c \|x\|_2\) for all \(x \in X\), show that there is a constant \(C\) such that \(\|x\|_2 \leq C \|x\|_1\) for all \(x \in X\).

**Solution:** Consider the map \(I : X_2 \to X_1\) given by \(Ix = x\). Then \(I\) is clearly linear from \(X_2 \to X_1\), and by the hypotheses, we have
\[
\|Ix\|_1 \leq c \|x\|_2 \quad \forall x \in X
\]
so \(I\) is bounded. But, we also have that \(I\) is bijective, and so by the Bounded Inverse Theorem, we have that \(I^{-1} : X_1 \to X_2\) is a bounded linear operator as well, i.e., there
exists a constant $C$ such that

$$\| I^{-1}x \|_2 \leq C \| x \|_1 \quad \forall x \in X.$$  

However, $I^{-1}x = x$ and so

$$\| x \|_2 \leq C \| x \|_1 \quad \forall x \in X.$$  

6. Let $X$ be the normed space whose points are sequences of complex numbers $x = \{ \xi_j \}$ with only finitely many non-zero terms and norm defined by $\| x \| = \sup_j |\xi_j|$. Let $T : X \to X$ be defined by

$$Tx = \left( \frac{1}{1} \xi_1, \frac{1}{2} \xi_2, \frac{1}{3} \xi_3, \ldots \right) = \left( \frac{1}{j} \xi_j \right).$$

Show that $T$ is linear and bounded, but $T^{-1}$ is unbounded. Does this contradict the Bounded Inverse Theorem?

**Solution:** It is pretty obvious that $T$ is linear. Since $x = \{ \xi_n \}$ and $y = \{ \eta_n \}$, then we have that

$$T(x + y) = \left\{ \frac{1}{j} (\xi_j + \eta_j) \right\} = \left\{ \frac{1}{j} \xi_j \right\} + \left\{ \frac{1}{j} \eta_j \right\} = Tx + Ty.$$  

The case of constants $\lambda$, is just as easy to verify, $T\lambda x = \lambda Tx$. It is also very easy to see that $T$ is bounded, since we have for any $x \in X$ that

$$\| Tx \|_X = \sup_{j \geq 1} \frac{|\xi_j|}{j} \leq \| x \|_X \sup_{j \geq 1} \frac{1}{j} \leq \| x \|_X.$$  

We now show that $T^{-1}$ is not bounded. First, note that $T^{-1}$ is given by

$$T^{-1}x = \{ j \xi_j \}.$$  

Now let $x = \{ \xi_j \}$ where $\xi_j = 0$ if $j \neq n$ and $1$ if $n = j$. Then we have that $\| x \|_X = 1$ and $T^{-1}x = \{ j \xi_j \} = (0, \ldots, 0, n, 0, \ldots)$ where the value $1$ occurs in the $n$th coordinate. This then implies that $\| T^{-1}x \|_X = n$. But, this is enough to show that the operator is not bounded. This doesn’t contradict the Bounded Inverse Theorem since $X$ is not complete.

7. Give an example to show that boundedness need not imply closedness.

**Solution:** Let $T : D(T) \to D(T) \subset X$ be the identity operator on $D(T)$, where $D(T)$ is a proper dense subspace of the normed space $X$. Then it is immediate that $T$ is linear and bounded. However, we have that $T$ is not closed. To see this, take $x \in X \setminus D(T)$ and a sequence $\{ x_n \} \in D(T)$ that converges to $x$, and then use the simple test for an operator to be closed that we gave in class.
8. Show that the null space of a closed linear operator $T : X \rightarrow Y$ is a closed subspace of $X$.

**Solution:** Recall that the null space of a linear operator is given by $\text{Null}(T) := \{ x \in X : T x = 0 \}$. Let $x \in \text{Null}(T)$ and suppose that $x_n \in \text{Null}(T)$ is a sequence such that $x_n \rightarrow x$. We need to show that $x \in \text{Null}(T)$.

If we have that $Tx_n \rightarrow y$, then since each $x_n \in \text{Null}(T)$ we have that $y = 0$. But since $T$ is closed, then we have that $Tx = y = 0$ and so $x \in \text{Null}(T)$.

9. Let $H$ be a Hilbert space and suppose that $A : H \rightarrow H$ is everywhere defined and

$$\langle Ax, y \rangle_H = \langle x, Ay \rangle_H \quad \forall x, y \in H.$$ 

Show that $A$ is a bounded linear operator.

**Solution:** It suffices to show that the graph of $A$ is closed. So suppose that we have $(x_n, Ax_n) \rightarrow (x, y)$. Then we need to show that $y = Ax$. To accomplish this last equality, it is enough to show that for all $z \in H$ that

$$\langle z, y \rangle_H = \langle z, Ax \rangle_H.$$ 

But, with this observation the problem becomes rather easy:

$$\langle z, y \rangle_H = \lim_n \langle z, Ax_n \rangle_H = \lim_n \langle Az, x_n \rangle_H = \langle Az, x \rangle_H = \langle z, Ax \rangle_H.$$ 

10. Let $\{T_n\}$ be a sequence of bounded linear transformations from $X \rightarrow Y$, where $X$ and $Y$ are Banach spaces. Suppose that $\|T_n\| \leq M < \infty$ and there is a dense set $E \subset X$ such that $\{T_n x\}$ converges for all $x \in E$. Show that $\{T_n x\}$ converges on all of $X$.

**Solution:** Let $\epsilon > 0$ be given. Pick any $x \in X$. Now choose $x_0 \in E$ such that $\|x - x_0\|_X < \frac{\epsilon}{3M}$, which will be possible since $E$ is dense in $X$. For this $x_0$ we have that $T_n x_0 \rightarrow y$. So there exists an integer $N$ such that if $n, m > N$ then

$$\|T_n x_0 - T_m x_0\|_Y < \frac{\epsilon}{3}.$$ 

Now, we claim that $\{T_n x\}$ is Cauchy in $Y$, and since $Y$ is a Banach space it is complete.
and so the result follows. To see that \( \{T_n x\} \) is Cauchy, when \( n, m > N \) we have

\[
\|T_n x - T_m x\|_Y = \|T_n x - T_n x_0 + T_n x_0 - T_m x_0 + T_m x_0 - T_m x\|_Y \\
\leq \|T_n x - T_n x_0\|_Y + \|T_n x_0 - T_m x_0\|_Y + \|T_m x - T_m x_0\|_Y \\
< \|T_n\| \|x - x_0\|_X + \|T_m\| \|x - x_0\|_X + \frac{\epsilon}{3} \\
< 2M \frac{\epsilon}{3M} + \frac{\epsilon}{3} = \epsilon.
\]