Fundamental Group

\[ \pi_1(X) = \text{group of homotopy classes of based paths in } X. \]

Will see: \[ X \simeq Y \implies \pi_1(X) \simeq \pi_1(Y) \]

Examples:

1. \( \mathbb{R}^3 \) - unknot \( \rightarrow \mathbb{Z} \)

2. \( \mathbb{R}^3 \) - unlink

   \[ a b a^{-1} b^{-1} : \]

3. \( \mathbb{R}^3 \) - Hopf link

   \[ a b a^{-1} b^{-1} : \]

   Is \( \pi_1 \) abelian?

- push these two strands in tandem around the left-hand circle to see triviality.
Formal Definitions

A path in a space $X$ is a map $I \rightarrow X$.

A homotopy of paths is a homotopy $f_t : I \rightarrow X$ such that $f_t(0)$ and $f_t(1)$ are independent of $t$.

Example. Any two paths $f_0, f_1$ in $\mathbb{R}^n$ with same endpoints are homotopic via straight-line homotopy:
\[ f_t(s) = (1-t)f_0(s) + tf_1(s) \]

Exercise. Homotopy of paths is an equivalence relation. \(\simeq\)

The composition of paths $f, g$ with $f(1) = g(0)$ is the path $f \circ g$:
\[ f \circ g(s) = \begin{cases} f(2s) & 0 \leq s \leq \frac{1}{2} \\ g(2s-1) & \frac{1}{2} \leq s \leq 1 \end{cases} \]

Exercise. $f_0 \simeq f_1$, $g_0 \simeq g_1 \Rightarrow f_0g_0 \simeq f_1g_1$

A loop is a path $f$ with $f(0) = f(1)$.

The fundamental group of $X$ (based at $x_0$) is the group of homotopy classes of loops based at $x_0$ under composition. Write $\pi_1(X, x_0)$. 
Prop: \( \pi_1(X, x_0) \) is a group.

Proof: Identity = constant loop

\[ \begin{array}{c}
\text{Associativity: } \\
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{f} \\
\text{g} \\
\text{h}
\end{array}
\begin{array}{c}
\text{f} \\
\text{g} \\
\text{h}
\end{array}
\begin{array}{c}
\text{f} \\
\text{f}
\end{array}
\end{array}
\end{array}
\end{array} \]

Inverses:

\[ \begin{array}{c}
\text{const} \\
\text{f} \\
\text{f}
\end{array} \]

\[ f(t) = f(1-t) \]

Prop: \( X = \text{path connected}, \ x_0, x_1 \in X \Rightarrow \pi_1(X, x_0) \cong \pi_1(X, x_1) \]

The isomorphism is not canonical!

Say \( X \) is simply connected if

1. \( X \) is path connected
2. \( \pi_1(X) = 1. \)

This terminology is explained by:

Prop: \( X \) is simply connected \( \iff \) there is a unique homotopy class of paths joining any two points of \( X. \)

Fact: Contractible \( \Rightarrow \) simply connected.
FUNDAMENTAL GROUP OF THE CIRCLE

Thm: \( \pi_1(S^1) \cong \mathbb{Z} \)

Proof outline: Given a loop \( f: I \to S^1 \), want to find a lift, that is:
\[
f: I \to \mathbb{R}
\]
such that \( f(0) = 0 \), \( pf = f \) ← ignore the international date line.

The map \( \pi_1(S^1) \to \mathbb{Z} \) is
\[
f \mapsto f(1)
\]

Well-definedness: existence/uniqueness of lifts
Multiplicativity: easy
Injectivity: homotopic loops have homotopic lifts
Surjectivity: easy

Remains to show loops lift uniquely and homotopies lift.

Idea: Cover \( S^1 \) by small pieces whose preimages in \( \mathbb{R} \) are unions of open intervals.
Given a loop/homotopy, cut it into pieces, lift piece by piece.

Proof thus follows from Lemma below.
Lemma: Given $F: Y \times I \to S^1$

$\tilde{F}: Y \times \{0\} \to \mathbb{R}$ lift of $F|_{Y \times \{0\}}$

$\exists! \tilde{F}: Y \times I \to \mathbb{R}$ lifting $F$, extending $\tilde{F}|_{Y \times \{0\}}$.

Path lifting: $Y = \{y_0\}$ Homotopy lifting: $Y = I$.

Proof (\(Y = \{y_0\}\) case): Write $I$ for $y_0 \times I$.

Cover $S^1$ by $\{U_x\}$ so that $\forall x$, $p^{-1}(U_x)$ is a disjoint union of open sets, each homeomorphic to $U_x$.

$F$ continuous $\Rightarrow$ can choose $0 = t_0 < t_1 < \ldots < t_m = 1$ so that $\forall i$, $F([t_i, t_{i+1}])$ is contained in some $U_{x_i}$; call it $U_i$.

Say $\tilde{F}$ defined on $[0, t_i]$, $\tilde{F}(t_i) \in U_i$, $pl_{U_i}: \tilde{U}_i \to U_i$ homeo.

Define $\tilde{F}$ on $[t_i, t_{i+1}]$ via $(pl_{U_i})^{-1} \circ F|_{[t_i, t_{i+1}]}$

Induct.

Exercise. Prove for general $Y$. \(\Box\)
Prop: $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$ for $X, Y$ path connected.

Cor: $\pi_1(T^2) \cong \mathbb{Z}^2$

Applications

Brouwer Fixed Point Theorem: Every $h: D^2 \to D^2$ has a fixed point.

Proof: Say $h(x) \neq x \ \forall \ x \in D^2$.
Can define $r: D^2 \to S^1$ via retraction.
Let $f_0 = \text{loop in } S^1 = \partial D^2$.
$f_t = \text{any homotopy to a point in } D^2$
$\Rightarrow r f_t = \text{homotopy in } S^1$ of $f_0$ to trivial loop.
Thus $\pi_1(S^1) = 1$. Contradiction.

Also:

Borsuk-Ulam theorem — for any $f: S^2 \to \mathbb{R}^2$, there is an antipodal pair $x, -x$ s.t. $f(x) = f(-x)$.

Ham Sandwich theorem.
Thrm: If we write $S^2$ as a union of 3 closed sets, at least one must contain a pair of antipodal points.
Fundamental Theorem of Algebra: Every nonconstant polynomial with coefficients in \( \mathbb{C} \) has a root in \( \mathbb{C} \).

**Proof:** Let \( p(z) = z^n + a_1z^{n-1} + \ldots + a_n \)

Define \( \tilde{p}(z) = z^n + t(a_1z^{n-1} + \ldots + a_n) \),

\[ \tau : \mathbb{C} \to S^1 \]

\[ \alpha \rightarrow \frac{\alpha}{|\alpha|} \]

\[ R > |a_1 + \ldots + a_n| + 1 \],

\[ f_{r,t}(s) : S^1 \to S^1 \]

\[ f_{r,t}(s) = \tau \circ p_t(r e^{2\pi i s}) \]

**Claim:** \( p_t \) has no roots on \( |z| = R \) for \( t \in I \).

\[ \Rightarrow f_{r,t}(s) \text{ defined.} \]

Thus the shaded path gives a homotopy from constant loop in \( S^1 \) to degree \( n \) loop \( r_z = 0 \).

Proof of Claim: For \( |z| = R \),

\[ |z^n| = R^n = R \cdot R^{n-1} > (|a_1| + \ldots + |a_n|) |z|^{n-1} \]

\[ > |a_1z^{n-1} + \ldots + a_n| \]

(But \( |\alpha| > |\beta| \Rightarrow \alpha + \beta \neq 0 \)).
**Induced Homomorphisms**

\[ \varphi : (X, x_0) \rightarrow (Y, y_0) \]
\[ \varphi_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0) \]
\[ [f] \mapsto [\varphi f] \]

**Functoriality**

1. \((\varphi \psi)_* = \varphi_* \psi_*\)
2. \(id_* = id\)

**Fact:** \(\varphi\) a homeomorphism \(\Rightarrow \varphi_* \) an isomorphism

**Proof:** \(\varphi_* \varphi_*^{-1} = ((\varphi \varphi^{-1})_*) = id_* = id\)

**Prop:** \(\pi_1(S^n) = 1\) for \(n \geq 2\).

**Proof:** \(S^n - pt \cong \mathbb{R}^n\), which is contractible.
   
   By Fact, suffices to show any loop in \(S^n\) is homotopic to one that is not surjective.

**Prop:** \(\mathbb{R}^2\) is not homeomorphic to \(\mathbb{R}^n\), \(n > 2\).

**Proof:** \(\mathbb{R}^n - pt \cong S^{n-1} \times \mathbb{R}\)
\[
\pi_1(S^{n-1} \times \mathbb{R}) \cong \pi_1(S^{n-1}) \times \pi_1(\mathbb{R})
\]
\[
\cong \begin{cases} 
\mathbb{Z} & n = 2 \\
1 & n > 2 
\end{cases}
\]

Apply Fact.
Prop: If \( \varphi: X \to Y \) homotopy equivalence, then \( \varphi_*: \pi_1(X, x_0) \to \pi_1(Y, \varphi(x_0)) \) isomorphism.

Proof: Let \( \psi: Y \to X \) homotopy inverse.
So \( \varphi \psi \cong \text{id} \).

Want \( \varphi_\psi \).
Remains to show: \( H_t: X \to X \) homotopy
\[ H_0 = \text{id} \]
\[ \Rightarrow (H_1)_*: \pi_1(X, x_0) \to \pi_1(X, H_1(x_0)) \]
an isomorphism.

We already know the path \( H_t(x_0) \) gives
\[ \pi_1(X, x_0) \xrightarrow{\cong} \pi_1(X, H_1(x_0)) \]
\[ f \mapsto H_t(x_0) f H_t(x_0) \]

But latter path \( \cong H_1 \circ f = (H_1)_*(f) \)

\[ H_1(x_0) \]
\[ \overbrace{\text{H}_1 \circ f} \]
\[ H_1(x_0) \]

So \( (H_1)_* \) an isomorphism. \( \Box \)
Prop: \( i: A \to X \) inclusion.
\( X \) retracts to \( A \) \( \Rightarrow i^* \) injective.
\( X \) deformation retracts to \( A \) \( \Rightarrow i^* \) isomorphism.

exercise. \( T^2 \) retracts to \( S^1 \).

In group theory, a retraction is a homomorphism
\( f: G \to H \), where \( H < G \), with \( f|_H = \text{id} \).
\( \Rightarrow G \cong H \times \ker f \).

**Free Groups and Free Products**

\( F_n = \{ \text{reduced words in } x_1^{\pm 1}, \ldots, x_n^{\pm 1} \} \)

- multiplication: concatenate, reduce.
- associativity: nontrivial!

\( G \ast H = \{ \text{reduced words in } G, H \} \)

\( \ast_{\alpha} G_\alpha \) similar = \( \{ g_1 \cdots g_m \mid g_i \in G_{\alpha_i}, \alpha_i \neq \alpha_{i+1}, g_i \neq \text{id} \} \)

example. Infinite dihedral group \( \mathbb{Z}/2\mathbb{Z} \ast \mathbb{Z}/2\mathbb{Z} \)

= symmetries of

**Properties**

1. \( G_\alpha \leq \ast G_\alpha \)
2. \( \cap G_\alpha = 1 \)
3. Any collection \( G_\alpha \to H \)
   extends uniquely to \( \ast G_\alpha \to H \)