Van Kampen's Theorem

\[ X = A \cup B \quad \text{A, B open, path connected.} \]
\[ A \cap B \quad \text{path connected.} \]
\[ x_0 \in A \cap B \quad \text{basepoint for } X, A, B, A \cap B. \]

The induced \( \pi_1(A) \to \pi_1(X) \) \& \( \pi_1(B) \to \pi_1(X) \)
extend to
\[ \phi : \pi_1(A) \ast \pi_1(B) \to \pi_1(X) \]

Denote \( i_A : A \to X, \ i_B : B \to X. \)

Let \( N = \text{normal subgroup of } \pi_1(A) \ast \pi_1(B) \)

\[ \text{generated by the } \ i_A(w) \ i_B(w)^{-1} \text{ for } w \in \pi_1(A \cap B). \]

**Theorem:**
1. \( \phi \) is surjective.
2. \( \text{Ker} \ \phi = N. \)

**Examples.**
1. \( \pi_1(S^1 \cup S^1) \cong F_2 \)
   
   \[
   \text{induction } \quad \Rightarrow \quad \pi_1(\bigvee_{n} S^1) \cong F_n
   \]
   
   \[
   \Rightarrow \quad \pi_1(\mathbb{R}^2 \setminus \text{n pts}) \cong \pi_1(\mathbb{R}^3 \setminus \text{unlink}) \cong F_n
   \]
   
   \[
   \pi_1(\text{graph}) \cong F_n
   \]
2. \( \pi_1(S^n) = \begin{cases} 1 \quad n > 2. \end{cases} \)
3. \( \pi_1(S^3 - (p,q)-\text{torus knot}) \cong \langle x, y \mid x^p = y^q \rangle \)

   \( \text{gluing two solid tori along an annulus.} \)
Van Kampen via Presentations.

\[ G_1 \cong \langle S_1 \mid R_1 \rangle \]
\[ G_2 \cong \langle S_2 \mid R_2 \rangle \]
\[ \Rightarrow G_1 \ast G_2 \cong \langle S_1 \cup S_2 \mid R_1 \cup R_2 \rangle \]

What is a presentation for \( \pi_1(A) \ast \pi_1(B) / N \)?

First, a given \( f \in \pi_1(A \cap B) \) gives two elements of \( \pi_1(A) \ast \pi_1(B) \):

\[ \pi_1(A) \xrightarrow{\pi_1(A) \ast \pi_1(B)} \pi_1(A) \xrightarrow{\pi_1(B)} \pi_1(B) \]

Call them \( f_A \) & \( f_B \).

Choose a generating set \( S \) for \( \pi_1(A \cap B) \).

Choose presentations:

\[ \pi_1(A) \cong \langle S_1 \mid R_1 \rangle \]
\[ \pi_1(B) \cong \langle S_2 \mid R_2 \rangle \]

so each \( S_i \) contains each \( f_A \) or \( f_B \) for \( f \in S \).

Then:

\[ \pi_1(A) \ast \pi_1(B) / N \cong \langle S_1 \cup S_2 \mid R_1 \cup R_2 \cup R \rangle \]

where \( R \) is the set of relations

\[ f_A = f_B \]

for \( f \in S \).
Proof 1) Let \( f: I \rightarrow X \) loop at \( x_0 \).

Choose \( 0 = S_0 < S_1 < \cdots < S_m = 1 \)

s.t. \( f|_{[S_i,S_{i+1}]} \) is a path in either \( A \) or \( B \);

call it \( f_i \).

\[ \forall i, \text{ choose path } g_i \text{ in } A \cap B \text{ from } x_0 \text{ to } f(S_i) \]

The loop

\((f_i \, g_i)(g_{i+1}f_{i+1}g_i)\cdots(g_{m-1}f_m)\)

is homotopic to \( f \), and is a composition of loops, \( \bar{\Phi} \), each in \( A \) or \( B \). \( \Rightarrow f \in \text{Im } \bar{\Phi} \).

2) A factorization of \( f \in \pi_1(X) \) is an element of \( \Phi^{-1}(f) \):

\[ f_i \cdots f_m \quad f_i \in \pi_1(A) \text{ or } \pi_1(B) \]

We showed in 1) that each \( f \) has a factorization.

Two factorizations are equivalent modulo \( N \) iff they differ by a sequence of moves:

(i) Combine \( [f_i][f_{i+1}] \rightarrow [f_i f_{i+1}] \)

if \( f_i, f_{i+1} \) lie both in \( \pi_1(A) \) or in \( \pi_1(B) \).

(ii) Regard \( [f_i] \in \pi_1(A) \) as \( [f_i] \in \pi_1(B) \)

if \( f_i \in \pi_1(A \cap B) \).

Let \( f_1 \cdots f_k, f'_1 \cdots f'_l \) factorizations of \( f \).
To show they are related by (i) \& (ii).
Choose a homotopy $I \times I \to X$ from one to the other.

Cut $I \times I$ into small rectangles, each mapping to $A$ or $B$, and so induced partitions of top & bottom edges are finer than those coming from the factorizations.

Push across one square at a time. Show the new factorization differs from old by (i) & (ii). E.g. two bottom-right squares.

Then rewrite $\alpha$ as $\alpha_1 \alpha_2$ (move (i)).

rewrite $\alpha_1$ as $\beta_1 \in \pi_1(B)$ (move (ii)).

Homotope $f_2 \beta_1 \in \pi_1(B)$ across square. etc. $\Box$
ATTACHING DISKS

X path connected, based at xo.
Attach 2-cell $D^2$ via $q: S^1 \to X$. 
\[ \to Y. \]
Choose path $f$ from $x_0$ to $q(S^1)$.
The loop $f q(S^1) \overline{f}$ is nullhomotopic in Y.
Let $N$ = normal subgroup of $\pi_1(X)$ generated
by this loop. Note: $N$ independent of $f$.

\textbf{Prop.} The inclusion $X \hookrightarrow Y$ induces a surjection
\[ \pi_1(X, x_0) \to \pi_1(Y, x_0) \]
with kernel $N$.

\textbf{Proof:} Choose $y \in \text{int}(D^2)$
Apply Van Kampen to $Y - y$, $Y - X$.
Note: $Y - y \simeq X$
$Y - X \simeq *$
\[(Y - y) \cap (Y - X) = \text{int}(D^2) - y \simeq S^1. \]

\textbf{Applications.} 1. $M_g$ = orientable surface of genus $g$.
$\pi_1(M_g) \cong \langle a_1, b_1, \ldots, a_g, b_g | [a_i, b_i] \cdots [a_g, b_g] = 1 \rangle$
\[ \Rightarrow M_g \neq M_h \text{ for } g \neq h \text{ as } \]
$\pi_1(M_g)^{ab} \cong \mathbb{Z}^{2g}$. 
(2) For any group $G$, there is a 2-dim cell complex $X_G$ with $\pi_1(X_G) \cong G$.

To do this, choose a presentation

$$G = \langle g_\alpha \mid r_\beta \rangle$$

$$X_G = \times S^1 \text{ with 2-cells attached along } r_\beta.$$