Classification of Covering Spaces

\[ \{ \text{based covers of } X \} \leftrightarrow \{ \text{subgroups of } \pi_1(X) \} \]

\[(\tilde{X}, \tilde{x}_0) \mapsto p_* (\pi_1(\tilde{X}, \tilde{x}_0)) \]

First step: find a cover corresponding to
trivial subgroup.

Theorem: \( X \) = CW-complex (or any path conn, locally path conn, semilocally simply conn.) Then \( X \) has a universal cover \( \tilde{X} \).

Proof: We construct \( \tilde{X} \) directly.

Points in \( \tilde{X} \) \( \leftrightarrow \) homotopy classes of paths from \( \tilde{X} \)
(simple connectivity)
\[ \leftrightarrow \] homotopy classes of paths from \( x_0 \)
(homotopy lifting)

So define:
\[ \tilde{X} = \{ [\tilde{f}] : \tilde{f} \text{ a path in } X \text{ at } x_0 \} \]

\[ p: \tilde{X} \rightarrow X \]
\[ [\tilde{f}] \mapsto \tilde{f}(1) \]
Topology on \( \tilde{X} \)

\[ U = \{ U \subseteq \tilde{X} : U \text{ path conn.}, \pi_1(U) \to \pi_1(\tilde{X}) \text{ trivial} \} \]

For \( U \in \mathcal{U} \), \( \tilde{f} \) with \( \tilde{f}(1) \in U \), define

\[ U[\tilde{f}] = \{ [\tilde{f} \cdot \eta] : \eta \text{ a path in } U, \eta(0) = \tilde{f}(1) \} \]

= open neighborhood of \([\tilde{f}]\) in \( \tilde{X} \).

exercise: The \( U[\tilde{f}] \) form a basis.

We now check the properties of a covering space.

- Continuity. \( p^{-1}(U) \) is a union of \( U[\tilde{f}] \)

- Path connectivity. Let \([\tilde{g}] \in \tilde{X} \).
  \[ J_t = \{ \tilde{f} \text{ on } [0,t] \}
  \text{ const. on } [t,1] \]
  is a path from \([\text{const}]\) to \([\tilde{g}]\).

- Simple connectivity. \( p_* \) injective, so suffices to show
  \[ p_* \pi_1(\tilde{X}) = 1 \]
  Let \( J \in \text{Im } p_* \Rightarrow J \) lifts to a loop.
  The lift of \( J \) is \( \{ [J_t] \} \)
  loop \[ [J_1] = [J_0] \]
  or \( [\tilde{g}] = [\text{const}] \)
  \[ \Rightarrow \tilde{f} = 1 \text{ in } \pi_1(X) \).
Coveting Space. Note: If \([g'] \in U[g]\) then \(U[g] = U[g']\)

Thus, for fixed \(U \in U\), the \(U[g]\) partition \(p^{-1}(U)\)

\(p: U[g] \rightarrow U\) homeomorphism since it gives a bijection of open sets

\(V[g] \subseteq U[g] \iff V \subseteq U\)

for \(V \in U\).

Theorem: For every \(H \subseteq \pi_1(X)\) there is a covering space \(\tilde{X}_H \rightarrow X\)

with \(p_* \pi_1(\tilde{X}_H, \tilde{x}_0) = H\).

Proof: We realize \(\tilde{X}_H\) as a quotient \(\tilde{X}_H = X/\sim:\)

\([g] \sim [g']\) if \(f(1) = f'(1)\)

and \([g; g'] \in H\).

Exercise: \(\sim\) is an equivalence relation.

Check \(\tilde{X}_H\) a covering space:

Say \([g] \sim [g']\) with \(f(1) = f'(1) \in U \in U\).

Then \([g; \eta] \sim [g'; \eta]\) for any path \(\eta\) in \(U\).

\(\Rightarrow U[g]\) identified with \(U[g']\).

Check \(p_* \pi_1(\tilde{X}_H) = H:\)

Let \(\tilde{x}_0 = [\text{const}]\).

\(f \in \text{Im } p_* \iff \{[g; f]\}\) a loop in \(\tilde{X}_H\)

\(\iff [g_0] \sim [g_1]\)

i.e. \([\text{const}] \sim [g]\)

\(\iff f \in H\).
To finish classification, need to show $\tilde{X}_H$ unique.

**Def.** Covering spaces $p_1: \tilde{X}_1 \to X$ and $p_2: \tilde{X}_2 \to X$ are isomorphic if there is a homeomorphism $f: \tilde{X}_1 \to \tilde{X}_2$ with $p_1 = p_2 f$ (i.e. $f$ preserves fibers).

**Prop.** Two path connected covering spaces $p_1: (\tilde{X}_1, \tilde{x}_1) \to X$ and $p_2: (\tilde{X}_2, \tilde{x}_2) \to X$ are isomorphic if and only if

\[ \text{Im}(p_1)_* = \text{Im}(p_2)_*. \]

**Proof:**

$\implies$ easy.

$\Leftarrow$

Lifting criterion $\implies$ lift $p_1$ to $\tilde{p}_1: (\tilde{X}_1, \tilde{x}_1) \to (\tilde{X}_2, \tilde{x}_2)$ with $p_2 \tilde{p}_1 = p_1.$

By symmetry $\implies \tilde{p}_2$ with $p_1 \tilde{p}_2 = p_2.$

Note $\tilde{p}_1 \tilde{p}_2$ is a lift of $p_2$:

$p_2 \tilde{p}_1 \tilde{p}_2 = p_1 \tilde{p}_2 = p_2.$

Unique lifting $+ \tilde{p}_1 \tilde{p}_2(\tilde{x}_2) = \tilde{x}_2 \implies \tilde{p}_1 \tilde{p}_2 = \text{id}.$

Symmetry: $\tilde{p}_2 \tilde{p}_1 = \text{id}.$

$\implies \tilde{p}_1$ a homeo.

**Cor.** Every subgroup of a free group is free.
Some Examples of Covering Spaces

\[ S^1 \times \mathbb{R} \rightarrow T^2 \]

\[ T^2 \overset{(x_m, x_n)}{\rightarrow} T^2 \]

Annulus \rightarrow Möbius strip

\[ S^2 \rightarrow \mathbb{RP}^2 \]

\[ \mathbb{C}^* \rightarrow \mathbb{C}^* \]

\[ \mathbb{C}^* \rightarrow T^2 \]

\[ \overrightarrow{\infty} \rightarrow \infty \]

\[ \overrightarrow{\infty} \rightarrow \infty \]

\[ \begin{array}{c}
\text{Diagram 1} \\
\text{Diagram 2} \\
\text{Diagram 3}
\end{array} \rightarrow \begin{array}{c}
\text{Diagram 4} \\
\text{Diagram 5}
\end{array} \]
THE FUNDAMENTAL THEOREM

Fix \( p : (\tilde{X}, \tilde{x}_0) \to (X, x_0) \)
\( H = p_* \pi_1(\tilde{X}, \tilde{x}_0) \)
\( N(H) = \text{normalizer in } \pi_1(X, x_0) \)
\( G(\tilde{X}) = \text{group of deck transformations.} \)

Say \( p \) is \underline{regular} if \( G(\tilde{X}) \) acts transitively on \( p^{-1}(x_0) \).

Regard \( \tilde{x}_0 \) as \([\text{const}]\)
Then \( p^{-1}(x_0) = \{ [\gamma] : \gamma \text{ a loop} \} \)

By lifting criterion, \( \exists \gamma \)
\( \exists \text{ deck trans taking } [\text{const}] \text{ to } [\gamma] \)
\( \iff p_* \pi_1(\tilde{X}, [\gamma]) = p_* \pi_1(\tilde{X}, [\text{const}]) \)
or \( \forall \gamma, p_* \pi_1(\tilde{X}, [\text{const}]) \gamma^{-1} = p_* \pi_1(\tilde{X}, [\gamma]) \)
i.e. \( \gamma \in N(H) \).

We thus have:
\( N(H) \to G(\tilde{X}) \)
\( \gamma \mapsto [\gamma] \)

Note: well-defined by uniqueness of lifts.

Prop: \( \tilde{X} \) regular \( \iff H \) normal.

Theorem: \( G(\tilde{X}) \cong N(H)/H \)

Both are exercises.
Covering Spaces via Actions

An action of a group \( G \) on a space \( Y \) is a homomorphism:
\[
G \to \text{Homeo}(Y)
\]
This is a covering space action if:
\[
\forall y \in Y \exists \text{ neighborhood } U \text{ with } \{ g(U) : g \in G \}
\]
all distinct, disjoint.

Fact: The action of \( G(\hat{X}) \) on \( \hat{X} \) is a covering space action.

Prop: \( Y = \) connected CW-complex
(or any path conn, locally path conn)
\( G \times Y \) via covering space action. Then:
(i) \( p : Y \to Y/G \) a regular covering space.
(ii) \( G \cong G(Y) \)

In particular:
\[ G \cong \pi_1(Y/G)/p_*\pi_1(Y) \]
- \( Y \) simply connected \( \Rightarrow \pi_1(Y/G) \cong G \).

Examples:
- \( \mathbb{Z} \times \mathbb{R} \to S^1 \)
- \( \mathbb{Z} \times \mathbb{R} \times I \to \text{Moebius strip} \)
- \( \mathbb{Z}^2 \times \mathbb{R}^2 \to T^2 \) - Klein bottle
- \( \mathbb{Z}/2\mathbb{Z} \times S^n \to \mathbb{R}P^n \)
- \( \mathbb{Z}/m\mathbb{Z} \times M_{m+n} \to M_{k+1} \)
**K(G,1) Spaces**

**Goal:** groups $\leftrightarrow$ spaces (up to homotopy equiv.)

homomorphisms $\leftrightarrow$ continuous maps (up to homotopy)

A $K(G,1)$ space is a space with fundamental group $G$
and contractible universal cover.

**Examples.** $S^1, T^2$ in general $\mathbb{Z}^n \leftrightarrow T^n$

What about $G = \mathbb{Z}/m\mathbb{Z}$?

$\mathbb{Z}/m\mathbb{Z}$ acts on $S^\infty = \text{unit sphere in } C^\infty$ via

$$ (z_i) \mapsto e^{2\pi ini} (z_i) $$

which is a covering space action.

(When $m=2$, quotient is $\mathbb{RP}^\infty$).

Why is $S^\infty$ contractible?

**Step 1:** $f_t(x_1, x_2, \ldots) = (1-t)(x_i) + t(0, x_1, x_2, \ldots)$

**Step 2:** Straight line projection to $(1,0,0,\ldots)$.

Later: Any $K(\mathbb{Z}/m\mathbb{Z})$ is $\infty$-dim!
Construction of $K(G,1)$ spaces

Prop: Every group $G$ has a $K(G,1)$

Proof: Define a $\Delta$-complex $EG$ with:

Ordered
$n$-simplices $\leftrightarrow (n+1)$-tuples
$[g_0, \ldots, g_n]$ \quad $g_i \in G$

To see $EG$ contractible, slide each $x \in [g_0, \ldots, g_n]$ along line segment in $[e, g_0, \ldots, g_n]$ from $x$ to $[e]$

(Note: This is not a deformation retraction since it moves $[e]$ around $[e, e]$.)

$G \rtimes EG$ by left multiplication.

Exercise: This is a covering space action.

$\sim \quad BG = EG/G$ is a $K(G,1)$.

This gives one $K(G,1)$, and it is always $\infty$-dim.

To study a group $G$, need a good $K(G,1)$, e.g. $K(PB_n,1) = G^n \setminus \Delta$. 
**Homomorphisms as Maps**

**Prop:** $X$ is connected CW-complex  
$Y = K(G,1)$  
Every homomorphism $\pi_1(X, x_0) \to G$ is induced  
by a map $(X, x_0) \to (Y, y_0)$.  
The map is unique up to homotopy fixing $y_0$.

This implies:

**Prop:** The homotopy type of a CW-complex $K(G,1)$  
is uniquely determined by $G$.

**Proof of 1st Prop:** Assume first $X$ has one 0-cell, $x_0$.

Let $\varphi : \pi_1(X, x_0) \to \pi_1(Y, y_0)$. Want $f : X \to Y$.

Step 0. $f(x_0) = y_0$.

Step 1. Each edge $e$ of $X$ is an element of $\pi_1(X, x_0)$. Define $f(e)$ via $\varphi$.

Step 2. Let $\Delta = 2$-cell with $\psi : \partial \Delta \to X^{(1)}$  
$f \psi$ null-homotopic, since $\varphi$ a homom.  
\[ \to \text{can extend } f \text{ to } \Delta. \]