Solution to problem 2

November 12, 2012

Solution. I will assume you know how to prove it’s a subspace.

Ok, so you have polynomials of degree at most 5 satisfying \( f(x) - 2f(x + 1) + f(x + 2) = 0 \). Let’s see where \( f(x) \to f(x) - 2f(x + 1) + f(x + 2) \) maps the term \( x^a \), where \( a = 0, 1, 2, 3, 4, \) or \( 5 \): we get that \( 1 \) and \( x \) get sent to \( 0 \); but \( x^2 \) gets sent to \( x^2 - 2(x+1)^2 + (x+2)^2 = 2 \), which is NOT \( 0 \); \( x^3 \) gets sent to \( x^3 - 2(x+1)^3 + (x+2)^3 = 6x + 6 \); \( x^4 \) gets sent to a degree 2 polynomial; and \( x^5 \) gets sent to a degree 3 polynomial.

So, if \( f(x) \) is a polynomial of degree \( d \), writing it as a linear combination of \( 1, x, x^2, x^3, x^4 \) and \( x^5 \), it’s clear that \( f(x) - 2f(x + 1) + f(x + 2) = 0 \) if and only if \( f \) has degree at most 1. Therefore, the space is 2 dimensional.