Instructions: You must supply your own paper. You will be allowed only a simple calculator – no programmable calculators. You have one hour to complete the exam.

1. Define the following terms.
   a. Say what it means for $X_1, \ldots, X_k$ to be independent random variables.
   b. Define the conditional expectation $\mathbb{E}(X|Y = y)$ in terms of the conditional probability density function $f(x|y)$, and then define $f(x|y)$.
   c. Define the moment generating function of a random variable $X$.
   d. Define the $m$th moment of a random variable $X$.
   e. Define the marginal probability density function for a random variable $X$, in terms of the joint probability density function for the pair $(X, Y)$.

2. a. Compute the moment generating function $M_X(t)$ for the random variable $X$ having the probability density function

$$f(x) = \begin{cases} 2x, & \text{if } x \in [0, 1]; \\ 0, & \text{otherwise}. \end{cases}$$

b. Using your answer from part a, compute the 3rd moment of $X$. Don’t just write down the answer – explain how you used moment generating functions to find it. (Note: You can easily check your answer, because the 3rd moment is easy to compute directly. This is not always the case, however, as mgf’s often provide a much easier way to find moments, than direct computation.)

3. Suppose that $(X, Y)$ is a 2D random variable with probability density function given by $f(x, y) = cx^2y$ when $(x, y)$ is confined to boundary and
interior of the triangle with vertices (0, 0), (1, 0), (0, 1); and suppose \( f(x, y) = 0 \) outside that triangle. Determine the constant \( c \).

4. Suppose you roll a 4-sided fair die (called a “D4”), and then flip two fair coins. Let \( X \) be the value of the roll (the numbers 1, 2, 3, 4 are printed on each side of the D4 – the value of your roll is the number printed on the bottom side), and let \( Y \) be the number of heads that you flipped.

   a. Determine the probability density function of the random variable \( Z = X + Y \).

   b. Determine the joint probability density function for \((X, Z)\). (One way to do this is to make a \(4 \times 3\) table of probabilities.)

   c. Determine the conditional expectation \( \mathbb{E}(X | Z = 4) \).

5. Prove that if \( X \) and \( Y \) are independent random variables, then

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V(X + Y) = V(X) + V(Y).
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