Homework 1, Math 3225

September 10, 2010

1. Recall that events $A_1, ..., A_k$ are independent if for every non-empty subset $S$ of \{1, ..., $k$\} we have

$$P(\cap_{s \in S} A_s) = \prod_{s \in S} P(A_s).$$

(1)

As a consequence of this, it turns out that this implies, and is equivalent to, the statement

**Claim.** For $i = 1, 2, ..., k$ we have that for any set $B$ in the $\sigma$-algebra generated by all the sets $A_j$, $j \neq i$,

$$P(A_i, B) = P(A_i)P(B).$$

In the special case $i = 1$ this would be saying that for any $B \in \sigma(A_2, ..., A_k)$ – i.e. $B$ is any set gotten by doing any number of intersections, unions and complements (which will turn out to be finite in number, since $k < \infty$) of the events $A_2, ..., A_k$ – we must have that $P(A_1, B) = P(A_1)P(B)$.

This equivalence is somewhat tricky to prove; and to give you a taste of what is involved, your first problem will be to prove the following: Suppose that $A, B, C,$ and $D$ are independent events (using the (1) definition above). Show that

$$P(A \cap (B \cup C \cup D)) = P(A)P(B \cup C \cup D).$$
2. Show that the map
\[ \varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \]
which sends
\[ (a, b) \mapsto a + \frac{(a + b - 2)(a + b - 1)}{2}, \]
is a bijection. Here, \( \mathbb{N} \) is the set of positive integers (i.e. Natural numbers). Hint: Draw the \( 5 \times 5 \) square of numbers \((a, b), 1 \leq a, b \leq 5\), and see where \((a, b)\) gets sent.

3. In a certain “pick 3” lottery a person selects a number from among 1000 possibilities \{000, ..., 999\}. Let us suppose each number is equally likely. Now suppose 500 people play the game, and each picks a number. From the lottery commission’s perspective, the worst thing that could happen is if the 500 people conspire and each picks a different number. If they do this, then the chance that at least one of them wins (and splits his winnings with the other 499 people he colludes with) is obviously 1/2. Now suppose the 500 people don’t conspire, and each picks his or her number independently of the other players. Show that the probability that at least one of the players picks the winning number is approximately \(1 - \frac{1}{\sqrt{e}}\). In working this problem, state what your sample space is, what the probability measure is, and any assumptions you make.