1. In this problem I will walk you through a proof that the sample variance is unbiased. Let

\[ S^2 = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n-1} \]

denote the sample variance, where we assume \( X_1, \ldots, X_n \) are i.i.d.

a. First show that \( E(S^2) = n\frac{E(X_1^2) - E(X_1)^2}{n-1} \).

b. Next, using the fact that \( V(X) = E(X^2) - E(X)^2 \), show that \( E(X_1^2) = V(X_1)/n + E(X_1)^2 \).

c. Finally, deduce that \( S^2 \) is unbiased.

2. Determine, with proof, a maximum likelihood estimate for the parameter \( \lambda \) in a Poisson random sample. That is to say, given a sample \( X_1, \ldots, X_k \) of i.i.d. random variables sampled from a Poisson distribution having pdf \( \lambda^n e^{-\lambda}/n! \), determine the maximum likelihood estimator \( \hat{\lambda} = \hat{\lambda}(X_1, \ldots, X_k) \).

3. Recall that the sample correlation coefficient \( r \) for a sample \( (X_1, Y_1), \ldots, (X_n, Y_n) \) is defined to be

\[ r = \frac{S_{XY}}{\sqrt{S_{XX}S_{YY}}} \]

where for sample variables \( (V_i, W_i) \) we define

\[ S_{VW} := \sum_i (V_i - \bar{V})(W_i - \bar{W}) \]

Now, the Cauchy-Schwarz inequality says the following.
Cauchy-Schwarz. Suppose that \( V_1, \ldots, V_n \) and \( W_1, \ldots, W_n \) are any complex numbers. Then,

\[
\left| \sum_{i=1}^{n} V_i W_i \right|^2 \leq \left( \sum_{i=1}^{n} |V_i|^2 \right) \left( \sum_{i=1}^{n} |W_i|^2 \right),
\]

where equality holds if and only if \( V_i = \lambda W_i \) for some complex number \( \lambda \).

Using Cauchy-Schwarz, prove that \(-1 \leq r \leq 1\), and then also that \(|r| = 1\) if and only if \( X \) and \( Y \) are linearly related – i.e. there exists \( \lambda_1, \lambda_2 \) such that \( Y_i = \lambda_1 X_i + \lambda_2 \).

4. Consider the following sample of fat content (in percentage) of \( n = 10 \) randomly selected hot dogs:

\[
25.2, 21.3, 22.8, 17.0, 29.8, 21.0, 25.5, 16.0, 20.9, 19.5.
\]

Assuming that these were selected from a normal population distribution, determine a 95% confidence interval of the population mean of fat content; and, determine a 90% confidence interval for the population variance of fat content.

5. In an experiment designed to measure the time necessary for an inspector’s eyes to become used to the reduced amount of light necessary for penetrant inspection, the sample average time for \( n = 9 \) inspectors was 6.32 seconds and the sample standard deviation was 1.65 seconds. It has previously been assumed that the average adaptation time was at least 7 seconds. Assuming that the adaptation time to be normally distributed, does the data contradict prior belief? Use the \( t \) test with \( \alpha = 0.1 \).