1. (10 points)
   a. Define what it means for a set $G$ to be a group.
   b. Define what it means for a mapping $\varphi$ from a group $G$ to a group $G'$ to be a homomorphism. Also, define what it means for $\varphi$ to be an isomorphism.
   c. Define what it means for a subgroup $H$ of a group $G$ to be a normal subgroup.

2. (20 points) Suppose that $G$ is an abelian group of odd order.
   a. Prove that the product of all the elements of $G$ equals the identity.
   b. Show that this is not true when $G$ has even order (it is sometimes true, but not always, when $G$ has even order) by producing a group of even order whose product of elements does not equal the identity.

3. (30 points) Suppose that $G$ is a group of order 15. In this problem we will prove that $G$ has an element of order 5 (in a round-about way):
   Break $G$ down into orbits under conjugation, where $a$ and $b$ lie in the same orbit if and only if $b = g^{-1}ag$ for some $g \in G$.
   a. Conclude that if $G$ has a non-trivial center $Z$, then $|Z| = 3, 5$ or 15. If $|Z| = 15$, then $G$ is abelian, and we are done (we proved in class that abelian groups of order divisible by $p$ always contain a $p$-cycle).
   b. Next, show that $|Z| \neq 5$. If it did, show that any element $c \in G$, such that $c \not\in Z$, has centralizer containing $c$ and containing $Z$. Prove that this is not possible.
   c. Next, show that if $|Z| = 3$, then there exists an element $c \in G$, $c \not\in Z$, whose centralizer $C(c)$ has order 5. As in part b, do this by observing that $Z$ and $c$ belong to $C(c)$. 

d. Show that if $|Z| = 1$, then one of the orbits of $G$ under conjugation has order equal to 3. Conclude that the stabilizer (centralizer) of this orbit is a 5-cycle.

4. (20 points) Show that if $\varphi : G \to G'$ is an isomorphism, and if $\psi : G' \to G$ is the inverse mapping of $\varphi$, then $\psi$ is also an isomorphism.

5. (20 points)
   a. Show that $ad \equiv bd \pmod{nd}$ if and only if $a \equiv b \pmod{n}$.
   b. Solve for $x$ in the equation
      $$35x \equiv 107 \pmod{9361}$$
      Hint: Add multiples of 9361 to both sides and cancel off the 5 and the 7.