Chapter 9

Graphs

9.1 A Gentle Introduction
Four Problems

The Bridges of Konigsberg

Three House-Three Utility

Four Color

Traveling Salesman
A Sample Problem

Among you, your buddy, two mothers, and two sisters, some people hug. There are no hugs between buddies, mothers, or sisters. The other 5 people tell you they all hugged different numbers of people. How many people did you hug?

Your buddy did not hug 4 people (otherwise no one hugged 0 people) → may assume your mother hugged 4

So your buddy’s mom hugged 0. If your buddy hugged 3, nobody hugged 1. If your buddy hugged 1, nobody hugged 3. → your buddy hugged 2 → you hugged 2.
9.2 Definitions and Basic Properties
**Graphs**

A graph is a pair of sets $V$ and $E$, where $V \neq \emptyset$ and each element of $E$ is a pair of elements of $V$.

Write $G = G(V, E)$.

For us, graphs are finite, that is, $|V|$ is finite.

The elements of $V$ and $E$ are called vertices and edges.

**Example.** $V =$ Facebook users

$E =$ Friendships
Graphs

We can represent graphs with pictures.

Example. Consider the graph $G(V,E)$ where

\[ V = \{a, b, c, d, e\} \]
\[ E = \{\{d, b\}, \{a, c\}, \{e, b\}, \{e, c\}, \{d, a\}\} \]

Can describe a graph with a picture instead of set notation.

Could also write $E = \{db, ac, eb, ec, da\}$.

We say $a$ is adjacent to $c$ and $d$, and $ac$ is incident to $a$ and $c$. 
**DEGREES**

The degree of a vertex $v$ is the number of edges incident to $v$. Write $\text{deg } v$.

If $\text{deg } v = 0$, we say $v$ is isolated.
Pseudographs

The following two phenomena are not allowed in a graph:

If we allow these, we get what is called a pseudograph.

Pseudographs are harder to write down with set notation, so we usually describe them with a picture.

**Example.** Vertices are web pages

   Edges are links
**Subgraphs**

A subgraph of a graph $G(V,E)$ is a graph $G(V',E')$ where $V' \subseteq V$ and $E' \subseteq E$

**Example.** \( \overline{a-b} \) is a subgraph of \( f \) but \( \overline{a-c} \) is not.

Also: Can delete any number of edges to get a subgraph. Can delete any number of vertices (and all incident edges) to get a subgraph.
Three Special Families

$C_n$: $n$-cycle

$K_n$: Complete graph

$K_{m,n}$: Complete bipartite graph
A bipartite graph is one whose vertex set can be partitioned into two sets $V_1$ and $V_2$ so that each edge joins an element of $V_1$ to an element of $V_2$.

**Fact.** A bipartite graph contains no triangles. More generally, a bipartite graph contains no odd cycles.
The Handshaking Lemma

**Proposition.** The sum of the degrees of the vertices of a pseudograph is an even number. Specifically:

\[ \sum_{v \in V} \deg v = 2|E| \]

**Handshaking Lemma.** The number of odd degree vertices of a pseudograph is even.

**Proof.**

\[ \sum_{v \in V} \deg v = \sum_{v \text{ even}} \deg v + \sum_{v \text{ odd}} \deg v \]

Revisit the hugging problem.
**The Handshaking Lemma**

**Problem.** A graph has 50 edges, 4 vertices of degree 2, 6 of degree 5, 8 of degree 4, all other vertices have degree 6. How many vertices does the graph have?

**Problem.** Out of 24 curling players, 78 pairs have played on the same team. Show that one has played on the same team as 7 others. Show that one has played on the same team with no more than 6 others.
Degree Sequence

Say $d_1, \ldots, d_n$ are the degrees of the vertices of a pseudograph, where $d_1 \geq d_2 \geq \cdots \geq d_n$. Then $d_1, \ldots, d_n$ is the degree sequence of the pseudograph.

\[ \sim 5, 4, 3, 3, 3, 3, 3, 3, 3, 2 \]
Graph Isomorphism


\begin{align*}
&\text{Example.} \\
&\text{Change of labels taking one to the other.} \\
&\text{In other words, two graphs are isomorphic if there is a} \\
&\text{that preserves adjacency and nonadjacency.} \\
&\text{a bijection} \\
&\text{The graphs } (G(V,E) \text{ and } G'(V',E') \text{ are isomorphic if there is} \\
&\text{GRAPH ISOMORPHISM}}
\end{align*}
GRAPH ISOMORPHISM

Which of the following pairs are isomorphic?
Invariants of Graphs

We can use the following “fingerprints” of graphs in order to tell if two graphs are different:

(i) Number of vertices
(ii) Number of edges
(iii) Degree sequence
   etc.

It is possible for two graphs to have the same degree sequence and be nonisomorphic:

{2, 2, 2, 1, 1}  {2, 2, 2, 1, 1}