A Note about Functions

Formally, a function $f: A \rightarrow B$ is a binary relation, s.t. for every $a \in A$ there is exactly one $b \in B$ s.t. $(a, b)$ is in the relation. 

[function $\leftrightarrow$ mapping]

Not a function!
Definitions: Let $f : A \rightarrow B$ be a function from $A$ to $B$.

1. The domain of $f$ is $A$.
2. The target of $f$ is $B$.
3. The image of $f$ is

\[ \text{Image } f = \{ b \in B \mid b = f(a) \text{ for some } a \in A \} \]

4. The function is **onto** or **surjective** if \( \text{Image } f = B \).

That is, every $b \in B$ has an $a \in A$ s.t. $f(a) = b$.

**Onto:** For any $b \in B$, $b = f(x)$ has a solution $x \in A$
5. The function is **one-to-one (1-1)** or **injective** iff different elements of A are mapped to different elements in B.

**One-to-one:** If \( f(a_1) = f(a_2) \Rightarrow a_1 = a_2 \)**
ONTO Examples:

1. Onto IR,
2. Not onto IR [image is [-1, 1)].
3. Onto 2
4. Not onto 2 (image is all evens).
5. Onto [given that \{peoples\} contains all initials in \{A, ..., Z\}].
one-to-one examples

1. \[ f(x_1) = f(x_2) \]
   \[
   \downarrow
   \]
   \[ x_1^2 - ux_1 + 5 = x_2^2 - ux_2 + 5 \]
   \[
   \Rightarrow x_1^2 - ux_1 = x_2^2 - ux_2 \]
   \[
   \Rightarrow x_1^2 - x_2^2 = ux_1 - ux_2 \]
   \[
   \Rightarrow (x_1 - x_2)(x_1 + x_2) = u(x_1 - x_2) \]

If we assume \( x_1 \neq x_2 \), we can divide by \( (x_1 - x_2) \)

\[
\Rightarrow x_1 + x_2 = u \]

\[ x_1 = 1, \ x_2 = 2, \] this is a valid solution \( \Rightarrow f(x_1) = f(x_2) \) does not imply \( x_1 = x_2 \)

\[
\Rightarrow \text{function is not one-to-one.} \]
2. \( f(x) = \sin(x) \).  
   Not one-to-one
   \( \sin(0) = \sin(\pi) = 0 \).

3. \( f(x) = x^7 \), not one-to-one.  
   \( f(x) = 0 \) for \( x \) in \([0,1]\).

4. \( f(x) = 2x \) is one-to-one.  
   \( 2x_1 = 2x_2 \Rightarrow x_1 = x_2 \).

5. \( f: \{ \text{people} \} \rightarrow \{ \text{A, ..., Z} \} \),  
   Not one-to-one
   \( f(\text{Alice}) = f(\text{Anna}) = A \).
Example:

\[ N = 21, 2, \ldots, n. \]

\[ f: \mathcal{P}(N) \rightarrow \{0,1\}^n \]

\[ \forall \text{ all binary strings} \]

\[ \forall \text{ all subsets of length } n. \]

We know that there is a one-to-one correspondence between them. Every subset is mapped to a unique string: At the i-th bit string has 1 if i is in the subset and 0 otherwise.
**Inverse Function**

\[
f : A \rightarrow B
\]

\[
f^{-1} : B \rightarrow A
\]

\[
a = f(x) = x^2.
\]

\[
\downarrow
\]

\[
f^{-1}(x) = \sqrt{x}.
\]

2. \[
f(x) = 3x + 7.
\]

\[
\begin{align*}
f^{-1}(x) &= \frac{x - 7}{3} \\
f \circ f^{-1}(x) &= x
\end{align*}
\]

Identity.

In our example:

\[
\begin{align*}
f \circ f^{-1}(x) &= \int \left[ \frac{x - 7}{3} \right] \\
&= \left[ 3 \cdot \frac{x - 7}{3} + 7 \right] \\
&= x - 7 + 7 = x
\end{align*}
\]
\[ f(x) = \ln (2x - 5) \]

\[ y = \ln (2x - 5) \]

\[ e^y = 2x - 5 \]

\[ 2x = e^y + 5 \]

\[ x = \frac{e^y + 5}{2} \]

\[ f^{-1}(x) = \frac{e^x + 5}{2} \]

\[ y = f^{-1}(x) = \frac{x}{x-1}. \]

\[ \frac{x}{x-1} = y \Rightarrow x = yx - y \]

\[ x - yx = y \Rightarrow x (1-y) = y \]

\[ x = \frac{y}{1-y} \Rightarrow f^{-1}(x) = \frac{x}{1-x}. \]