Counting Rules

\[ |A \cup B| = |A| + |B| - |A \cap B| \]

\(|A| + |B|\) counts \(|A \cap B|\) twice, so need to subtract \(|A \cap B|\).

\[ |A \setminus B| = |A \cap B^c| + |B \cap A^c| \]

\(|A \setminus B| \geq\]

\[ |A| + |B| - |U| \]

We get equality if \(U = A \cup B\).
More problems

1. Solution:

\[ |U| = 80, \ |A| = 20, \ |B| = 15. \]
\[ |A \cap B| = 10. \]

What is \( |A \cup B| \)?

- By inclusion-exclusion rule:

\[ |A \cup B| = |A| + |B| - |A \cap B| \]
\[ = 20 + 15 - 10 = 25. \]

What is \( |(A \cup B)^c| \)?

\[ |(A \cup B)^c| = |U| - |A \cup B| = 80 - 25 \]
\[ = 55. \]

From previous section.
More problems

2. Solution:

|U| = 1000, |A| = 800, |B| = 500

|A \cap B| = 325

What is \((A \cup B)^c\)?

First, \(|A \cup B| = |A| + |B| - |A \cap B|\)

= 800 + 500 - 325 = 975.

Next:

\(|(A \cup B)^c| = |U| - |A \cup B| =

= 1000 - 975 = 25.
**Rolling Dice**

We have two dice, each with 6 faces.

\[ U_1 = \{1, 2, \ldots, 6\}, \]
\[ U_2 = \{1, 2, \ldots, 6\}. \]

\[ U_1 \times U_2 = \text{set of all 36 results.} \]
\[ = \{(1,1), (1,2), \ldots, (6,3), (6,6)\} \]

How many results we have, s.t.
first die does not show "1", and
second die does not show "2"?

**Solution:**

\[ A \subseteq U_1 \times U_2, \text{ all results, where first die shows } "1". \]
\[ A = \{ (1,1), (1,2), (1,3), (1,4), (1,5) \} \]

\[ B \subseteq U_{1, x U_{2}}: \text{all results with second die shows "2"}. \]

\[ B = \{ (1,2), (2,2), (3,2), (4,2), (5,2) \} \]

What is \( A \cap B \)?

\[ A \cap B = \{ (1,2) \}. \]

\[ |U_{1, x U_{2}}| = 36, \ |A| = 6, \ |B| = 6, \ |A \cap B| = 1 \]

\[ |A \cup B| = |A| + |B| - |A \cap B| = 6 + 6 - 1 = 11 \]
\[(A \cup B)^c = |U_1 \times U_2| - |A \cup B| \]
\[= 36 - 11 = 25.\]

Three dice

Next, we have 3 dice, and we would like to exclude all results with "1" at first die, "2" at second die, "3" at third die.

Here we have

- \(U_1 = U_2 = U_3 = \{1, 2, \ldots, 6\}\).
- \(A = \{ (1, x, y) \mid x \in U_2, y \in U_3 \}\), \(|A| = 36\)
- \(B = \{ (2, x, y) \mid 2 \in U_1, y \in U_3 \}\), \(|B| = 36\)
- \(C = \{ (2, x, 3) \mid 2 \in U_1, x \in U_2 \}\), \(|C| = 36\)
\[ A \cap B = \{ (1, 2, y) \mid y \in U_3 \} \]
\[ A \cap C = \{ (1, x, 3) \mid x \in U_2 \} \]
\[ B \cap C = \{ (2, 2, 3) \mid z \in U, z \} \]

\[ |A \cap B| = |A \cap C| = |B \cap C| = 6. \]

We are almost ready to obtain \[ |A \cup B \cup C| \].
\[(A \cup B \cup C) = \left| A \right| + \left| B \right| + \left| C \right| - \left| A \cap B \right| - \left| A \cap C \right| - \left| B \cap C \right| + \left| A \cap B \cap C \right|\]

Something is still missing.

In the subtraction of all intersection pairs \((A \cap B), \left| A \cap C \right|, \left| B \cap C \right|\), we overcounted \(\left| A \cap B \cap C \right|\).

Indeed, \(\left| A \right| + \left| B \right| + \left| C \right|\) (over)counts each pair \((A \cap B), \left| A \cap C \right|, \left| B \cap C \right|\) twice, and the triple \((A \cap B \cap C)\) is counted three times.

Each such "position" has to be counted exactly once!
\[ |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \]

In our dice problem we have:

\[ |A \cup B \cup C| = 3^3 - 3 \cdot 6 + 1 = 91 \]

\[ |(1, x, u, x, u_3)| - |A \cup B \cup C| = 6^3 - 91 = 125. \]

This is number of ordered triples with no \((1, x, y), (2, 2, y), (2, y, 3).\)
Inclusion - Exclusion problems

1. solution:

\[ |U| = 100, \quad |A| = 18, \quad |B| = 40, \quad |C| = 20, \quad |A \cap B| = 12, \quad |A \cap C| = 5, \quad |B \cap C| = 4, \quad |A \cap B \cap C| = 3. \]

What is \( |A \cup B \cup C| \)?

\[ |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| = 18 + 40 + 20 - 12 - 5 - 4 + 3 = 60. \]
What is \((A \cup B \cup C)^c\)?

\[(A \cup B \cup C)^c = (U) - |A \cup B \cup C| = 80 - 60 = 20.\]

How many only like Subway?

- This corresponds to \(C \setminus (A \cup B)\).

**Observation:**

\[C \setminus (A \cup B) = (A \cup B \cup C) \setminus (A \cup B).
\Rightarrow |C \setminus (A \cup B)| = |A \cup B \cup C| - |A \cup B|.

(since \(A \cup B \subseteq A \cup B \cup C\))
\[
\Rightarrow |C \setminus (A \cup B)| = |A \cup B \cup C| - |A \cup B|
\]
\[
= 60 - [|A| + |B| - |A \cap B|]
\]
\[
= 60 - [18 + 40 - 12]
\]
\[
= 14.
\]
Inclusion-Exclusion problems

How many integers between 1 and 500 are:

(a) not divisible by 2?

- \( A = \{ \text{all integers } 1 \leq n \leq 500, \text{ s.t. } 2 \nmid n \} \)

\[ |A| = \frac{500}{2} = 250. \]

\[ \Rightarrow |A^c| = 500 - 250 = 250. \]

(b) divisible by 2 or 3?

\[ B = \{ \text{all } n \leq 500 \text{ is an integer, s.t. } 3 | n \} \]

\[ A \cap B = \{ \text{all } n \leq 500 \text{ is an integer, s.t. } n \in \{2, 3 \} \} \]

since \( \gcd(2, 3) = 1 \), all integers divisible by 2 and 3 are all integers divisible by 6.
\[
|A \cup B| = |A| + |B| - |A \cap B| \\
= \frac{500}{2} + \frac{\sqrt{500}}{3} - \frac{\sqrt{500}}{6} \\
= 250 + 166 - 83 = 333.
\]

(c) Divisible by 2, 3, or 7.
\[
\mathcal{C} = \mathbb{P}(1 \leq n \leq 500 \text{ is an integer, } 5 \mid n). \\
|\mathcal{C}| = \left\lfloor \frac{500}{5} \right\rfloor,
\]
\[
|A \cap B| = \mathbb{P}(1 \leq n \leq 500 \text{ is an integer, } 6 \mid n) \\
|A \cap C| = \mathbb{P}(1 \leq n \leq 500 \text{ is an integer, } 14 \mid n) \\
|B \cap C| = \mathbb{P}(1 \leq n \leq 500 \text{ is an integer, } 21 \mid n).
\]
\[
|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.
\]
\[ A \cap B \cap C = \begin{cases} \frac{500}{8} + \frac{500}{3} + \frac{500}{7} & \left\lfloor \frac{500}{6} \right\rfloor - \left\lfloor \frac{500}{10} \right\rfloor - \left\lfloor \frac{500}{21} \right\rfloor + \left\lfloor \frac{500}{12} \right\rfloor \\ 2.37 + \ln 11 \end{cases} \]

\[ \Rightarrow (A \cup B \cup C) = 250 + 166 + 71 - \left[ 83 + 35 + 23 \right] + 11 \]

\[ = 357. \]
The addition rule

pairwise disjoint sets.

\[ |A_1 \cup A_2 \cup A_3 \cup A_4| = |A_1| + |A_2| + |A_3| + |A_4| \]

*(How many ways to roll a 6 with 2 dice?)*

- \[ \{(3,3), (2,4), (4,2), (1,5), (5,1)\} \]

\[ \Rightarrow 5 \text{ different ways.} \]
The Multiplication Rule

A set of cardinality $n$ has $2^n$ subsets.

Proof:

Any subset can be mapped to a bit string

\[ \underbrace{p_1, p_2, p_3, \ldots, p_n} \]

$n$ spots

The $i$-th bit is 1 if the subset contains the $i$-th element, and 0 otherwise. Overall, there are

\[ 2 \times 2 \times 2 \times \cdots \times 2 = 2^n \]

$n$ times

Choices.

\[ \Rightarrow \text{each choice corresponds to a unique subset.} \]
Problem: How many numbers in the range 1000-9999 do not have any repeated digits?

Solution: There are 9 possibilities for the first (leftmost) digit. The next digit has to be different than the first, but can be 0 ⇒ 9 possibilities. Third digit has to be different than the first 2 ⇒ 8 possibilities. Fourth has 7 possibilities ⇒ Overall: $9 \times 9 \times 8 \times 7$ possibilities.
Problem:

A license plate consists of four letters followed by three digits 0-9. How many different license plates can we make?

Solution:

There are 26 possibilities for first letter, 26 for second, third, and fourth \( \Rightarrow 26^4 \) possibilities for 4 first letters.

Similarly, there are \( 10^3 \) possibilities for the 3 digits.

\( \Rightarrow \) Overall, \( 26^4 \times 10^3 \) possibilities.