Math 4318 - Spring 2011
Homework 4

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in only problems 2, 4, 6, 11, 12, 13, 14, 15. Due: In class on March 10

1. If \( \{f_n\} \) and \( \{g_n\} \) are two sequences of functions that converge uniformly on a set \( D \subset \mathbb{R} \) then prove that \( \{f_n + g_n\} \) converges uniformly on \( D \) too. If the sequences are bounded (that is each \( f_n \) and \( g_n \) is bounded) then \( \{f_ng_n\} \) also converges uniformly on \( D \). Show that it is necessary to assume that \( f_n \) and \( g_n \) are bounded in order to conclude that \( \{f_ng_n\} \) converges uniformly.

2. Let 
\[
f_n(x) = \frac{x}{1 + nx^2}
\]
be a sequence of functions. Show that \( \{f_n\} \) converge uniformly to some function \( f \) and that
\[
f'(x) = \lim_{n \to \infty} f_n'(x)
\]
for all \( x \neq 0 \), but that the equality is not true for \( x = 0 \).

3. Let \( \mathcal{P} \) be the set of polynomials thought of as functions on \([0,1]\). So \( \mathcal{P} \subset C^0([0,1]) \). The uniform norm \( \| \cdot \|_\infty \) is a norm on \( \mathcal{P} \). Is \( (\mathcal{P}, \| \cdot \|_\infty) \) a Banach space? Hint: Think about the Weierstrass theorem.

4. Let \( f \in C^0([0,1]) \). Show that if
\[
\int_0^1 x^nf(x) \, dx = 0
\]
for all non-negative integers \( n \) then \( f(x) = 0 \).

Hint: Think about the Weierstrass theorem and try to show that \( \int_0^1 f^2(x) \, dx = 0 \).

5. Suppose that \( \{f_n\} \) is an equicontinuous sequence of functions on a compact set \( D \subset \mathbb{R} \) and that the sequence converges point-wise to \( f \). Show that \( \{f_n\} \) converges uniformly to \( f \) on \( D \).

6. Given two functions \( f, g \in \mathcal{R}([a,b]) \) define the \( L^2 \)-inner product to be
\[
\langle f, g \rangle = \int_a^b f(x)g(x) \, dx.
\]

Why is this not an inner product on \( \mathcal{R}([a,b]) \)? Show that this does give an inner product on the set of continuous functions \( C^0([a,b]) \). (Notice that this also gives a norm on \( C^0([a,b]) \) by \( \|f\|_2 = \sqrt{\langle f, f \rangle} \).)

7. Using the notation from the previous problem show that given any \( f \in \mathcal{R}([a,b]) \) there is a sequence of functions \( \{f_n\} \) in \( C^0([a,b]) \) such that \( \lim_{n \to \infty} \|f - f_n\|_2 = 0 \). Is \( C^0([a,b]) \) with the \( L^2 \)-norm complete? Hint: Of course you will have proven the result if you can show that for every \( \epsilon > 0 \) there is a continuous function \( g \) such that \( \|f - g\|_2 < \epsilon \). To show this use the Riemann integral (instead of the Darboux integral) and for a well chosen partition \( \mathcal{P} = \{x_0, \ldots, x_n\} \) let
\[
g(x) = \frac{x_i - x}{\Delta x} f(x_{i-1}) + \frac{x - x_{i-1}}{\Delta x} f(x_i).
\]
8. Given a function \( f \in \mathcal{R}([a, b]) \) show there is a sequence of polynomials \( p_n \) such that \( \lim_{n \to \infty} \| f - p_n \|_2 = 0 \).
   Hint: Use the previous problem and Weierstrass.

   Let \( V \) be a vector space and \( \| \cdot \|_a \) and \( \| \cdot \|_b \) be two norms on \( V \). We say the norms are **equivalent** if there are positive constants \( C \) and \( C' \) such that
   \[
   C \| \cdot \|_a \leq \| \cdot \|_b \leq C' \| \cdot \|_a.
   \]

9. Consider the vector space \( V = \mathbb{R}^2 \). Let \( \| (x, y) \| = \sqrt{x^2 + y^2} \) and let \( \| (x, y) \|_1 = |x| + |y| \).
   These are two norms on \( V \) (you do not have to show they are norms). Show that these two norms are equivalent.

**Interesting fact:** On a finite dimensional vector space any two norms are equivalent.

10. If \( \| \cdot \|_a \) and \( \| \cdot \|_b \) are equivalent norms on \( V \) then a sequence \( \{ v_n \} \) converges to \( v \) in the norm \( \| \cdot \|_a \) if and only if it converges to \( v \) in the norm \( \| \cdot \|_b \).

11. If \( \| \cdot \|_a \) and \( \| \cdot \|_b \) are equivalent norms on \( V \) then a sequence \( \{ v_n \} \) is Cauchy in the norm \( \| \cdot \|_a \) if and only if it is Cauchy in the norm \( \| \cdot \|_b \). (Together with the previous problem we see that \((V, \| \cdot \|_a)\) is a Banach space if and only if \((V, \| \cdot \|_b)\) is a Banach space when the norms are equivalent.)

12. Let \( \| \cdot \|_\infty \) be the sup-norm on \( C^1([a, b]) \). Is \((C^1([a, b]), \| \cdot \|_\infty)\) a Banach space? In class we saw that \((C^1([a, b]), \| \cdot \|_{C^1})\) is a Banach space (recall \( \| f \|_{C^1} = \| f \|_\infty + \| f' \|_\infty \)). Are the norms \( \| \cdot \|_\infty \) and \( \| \cdot \|_{C^1} \) equivalent on \( C^1([a, b]) \)?

13. Consider \( \mathcal{R}([a, b]) \) with the sup norm \( \| \cdot \|_\infty \) and \( C^0([a, b]) \) with the sup norm. Define a function
   \[
   I : \mathcal{R}([a, b]) \to C^0([a, b])
   \]
   by \( I(f)(x) = \int_a^x f(s) \, ds \). Show that \( I \) is a uniformly continuous function (that is given any \( \epsilon > 0 \) there is a \( \delta > 0 \) such that for all \( \| f - g \|_\infty < \delta \) we have \( \| I(f) - I(g) \|_\infty < \epsilon \)).

14. With the notation from the last problem assume that \( \{ f_n \} \) is a bonded sequence in \( \mathcal{R}([a, b]) \) with the sup norm. Show that \( \{ I(f_n) \} \) has a convergent subsequence (that is converges uniformly to some function \( f \) on \([a, b])\).

15. Let \( \{ f_n \} \) be a sequence of functions in \( C^1([a, b]) \) that are bounded in the norm \( \| \cdot \|_{C^1} \). Show that there is a subsequence that converges to a function \( f \) in the \( \| \cdot \|_\infty \) norm.

16. Suppose that \( f \in C^1([a, b]) \) and that \( f(a) = 0 \). Prove that
   \[
   \| f \|_\infty \leq \sqrt{b - a} \| f' \|_2
   \]
   where \( \| \cdot \|_2 \) is the \( L^2 \)-norm defined in Problem 6.
   Hint: Use the fundamental theorem of calculus and the Cauchy-Schwartz inequality.

This problem gives the idea behind the famous Sobolev embedding theorems. For example it does not take too much more work to show that if we set \( \| f \|_{1,2} = \| f \|_2 + \| f' \|_2 \) then this is a norm on \( C^1([a, b]) \) and there is some constant \( C \) such that \( \| f \|_\infty \leq C \| f \|_{1,2} \). In other words, the averages (that is integrals) of a function and its derivative control the pointwise values of the function. Or more precisely the identity map from \( C^1([a, b]) \) with the \( \| \cdot \|_{1,2} \) norm to \( C^1([a, b]) \) with the sup norm is continuous.