1) Let $f : [a, b] \to \mathbb{R}$ and suppose that there is some $M$ such that $|f'(x)| \leq M$. Prove using the definitions that $f$ is Lipschitz and continuous on $[a, b]$. 

**Solution:** For any $x, y$ in $[a, b]$ the Mean Value Theorem says there is some $c$ between $x$ and $y$ such that

$$|f(x) - f(y)| = |f'(c)||x - y| \leq M|x - y|.$$ 

So $f$ is Lipschitz with Lipschitz constant $M$. We know Lipschitz functions are continuous, but since we have to establish everything from definitions here is the proof. Given $x \in [a, b]$ and $\epsilon > 0$ let $\delta = \epsilon/M$, then if $y \in [a, b]$ and $|x - y| < \delta$ we see that

$$|f(x) - f(y)| \leq M|x - y| < M\delta = M(\epsilon/M) = \epsilon.$$ 

So $f$ is continuous.

2) Assuming that $f'$ exists on $[a, b]$ and $\lim_{x \to c} f'(x) = L$ for some $c \in (a, b)$, prove that $f'$ is continuous at $c$.

**Solution:** Since $c$ is a cluster point of $[a, b]$, $f'(x)$ being continuous at $c$ means that $\lim_{x \to c} f'(x) = f'(c)$. Thus we must show that $L = f'(c)$. If $L \neq f'(c)$ then let $\epsilon = \frac{|f'(c) - L|}{2}$. Since $\lim_{x \to c} f'(x) = L$ there is some $\delta > 0$ such that $|f'(x) - L| < \epsilon$ whenever $0 < |x - c| < \delta$ and $x \in [a, b]$. By making $\delta$ smaller if necessary we can assume that $c - \delta$ or $c + \delta$ is in $[a, b]$. Assuming the later we have for $c < x < c + \delta$ that

$$|f'(x) - f'(c)| \geq |L - f'(c)| - |f'(x) - L| > |L - f'(c)| - \epsilon = \frac{|f'(c) - L|}{2} = \epsilon > 0.$$ 

Thus $f'((c, c + \delta])$ is disjoint from $(f'(c) - \epsilon, f'(c) + \epsilon)$, but this contradicts the intermediate value theorem for derivatives.

3) Let $f : [a, b] \to \mathbb{R}$ be an integrable function with $f(x) \geq 0$ for all $x \in [a, b]$.

a) If $f$ is continuous at $c \in (a, b)$ and $f(c) > 0$ show that

$$\int_a^b f(x) \, dx > 0.$$ 

**Solution:** Since $f$ is continuous at $c$ there is some $\delta > 0$ such that for all $x \in [a, b]$ with $|x - c| < \delta$ we have $|f(x) - f(c)| < \frac{f(c)}{2}$. We can moreover assume that $\delta$ is small enough so that $I = (c - \delta, c + \delta) \subset [a, b]$. Now let $\chi_I$ be the characteristic function of $I$ (that is $\chi_I(x) = 1$ if $x \in I$ and zero otherwise). Then we know that for $|x - c| < \delta$ we have $f(x)/2 > f(c)/2$ Thus the function $g(x) = \frac{f(c)}{2}\chi_I$ satisfies $g(x) \leq f(x)$ for all $x$. Now

$$\int_a^b f(x) \, dx \geq \int_a^b g(x) = \int_{c-\delta}^{c+\delta} \frac{f(c)}{2} \, dx = \frac{f(c)}{2}(2\delta) > 0.$$ 

(You could also prove this using the definition of integral, but choosing partitions appropriately. For example show that the lower Darboux integral is greater than zero.)

b) If the set $C = \{x \in [a, b] : f(x) = 0\}$ has measure zero show that

$$\int_a^b f(x) \, dx > 0.$$
Solution: Since \( f \) is integrable we know the set \( D \) of points where \( f \) is discontinuous has measure zero. If \([a, b] - D \) had measure zero then so would \([a, b] = D \cup ([a, b] - D) \) (since the union of two sets of measure zero have measure zero). But it is easy to see that \([a, b]\) does not have measure zero (argument below). Thus \([a, b] - D \) does not have measure zero. From this we can conclude that \([a, b] - D \) is not a subset of \( C \) (since subsets of measure zero have measure zero). But since \([a, b] - D \) is the set of points where \( f \) is continuous it is clear that there is a point \( c \in (a, b) \) where \( f \) is continuous and not in \( C \), that is \( f(c) > 0 \). Thus we are done by part a).

Proof that \([a, b]\) does not have measure zero. Now suppose \( \{U_i\} \) is an cover of \([a, b]\) by open intervals. Since \([a, b]\) is compact there are a finite number of \( U_i \) that cover \([a, b]\), say \( U_1, \ldots, U_k \). We can assume that each \( U_i \) intersects \([a, b]\) (or else we could through it out and still have a cover). It is clear that \( U = \bigcup_{i=1}^k U_i \) is connected (if not then since \([a, b]\) is connected it would be in one of the components of \( U \) and thus there would be some \( U_i \) that don’t intersect \([a, b]\)). Thus \( U \) is an open interval that contains \([a, b]\) it is clear that the length of \( U \) is bigger than the length of \([a, b]\), that is \( b - a \). Thus the total length of the \( U_i \) is bigger than \( b - a \) and hence the total length of the \( U_i \) is bigger than \( b - a \). In particular we cannot find a cover of \([a, b]\) with total length less than, say, \( \frac{1}{2}(b - a) \). So \([a, b]\) does not have measure zero.

4) Let \( f : [0, 1] \to \mathbb{R} \) be the function that is 0 for all irrational numbers and \( f(x) = x \) for all rational numbers. Prove that \( f \) is not integrable. Hint: Show that the upper and lower Darboux integrals cannot be the same. Specifically show that any upper sum is bounded below by \( \frac{1}{2} \).

Solution: If \( \mathcal{P} \) is any partition of \([0, 1]\) then notice since each interval in the partition contains an irrational number we know that the minimum of \( f \) on each of these intervals is zero. Thus

\[
L(f, \mathcal{P}) = 0
\]

for all \( \mathcal{P} \) and so \( \int_0^1 f(x) \, dx = 0 \). Now if \( I = [x_{i-1}, x_i] \) is an interval of the partition \( \mathcal{P} \) then there is a sequence of rational numbers approaching \( x_i \) and thus the maximum of \( f \) on \( I \) is \( x_i \). Noting that \( \frac{x_i + x_{i-1}}{2} < x_i \) (because \( x_i > x_{i-1} \)) we have

\[
U(f, \mathcal{P}) = \sum_{i=1}^n x_i(x_i - x_{i-1}) \geq \sum_{i=1}^n \frac{1}{2}(x_i + x_{i-1})(x_i - x_{i-1}) = \frac{1}{2} \sum_{i=1}^n x_i^2 - x_{i-1}^2 = \frac{1}{2}(x_n^2 - x_0^2) = \frac{1}{2}(1 - 0) = \frac{1}{2}.
\]

Thus for any partition \( \mathcal{P} \) the upper sum of \( f \) is bounded below by \( \frac{1}{2} \) and hence

\[
\int_0^1 f(x) \, dx \geq \frac{1}{2}.
\]

In particular the upper and lower sums are not the same and hence \( f \) is not integrable.

5) Answer the following questions True or False. Circle either T or F to indicate your answer. You do not need to justify your answer.

I am providing reasons for the answers but you do not need to do so.
1. If $|f|$ is integrable on $[a, b]$ then $f$ is integrable on $[a, b]$.
   
   **F** If $f$ is 1 for irrational and $-1$ for rational numbers on $[0, 1]$ then $|f|$ is a constant function and hence integrable, but $f$ is discontinuous everywhere so it is not integrable.

2. If $f$ is not integrable on $[a, b]$ then there are partitions $P$ and $Q$ of $[a, b]$ such that $L(f, Q) > U(f, P)$
   
   **F** The upper sum is always larger than the lower sum for any partition.

3. If a function is differentiable on an open interval $I$ then it is continuous on $I$.
   
   **T** A theorem from class.

4. Sets of measure zero must be countable.
   
   **F** Countable sets have measure zero, but there are uncountable sets (like the middle thirds Cantor set) that also have measure zero.

5. If a function if differentiable on an open interval $I$ then its derivative is continuous on $I$.
   
   **F** The function $f(x) = x^2 \sin \frac{1}{x}$ for $x \neq 0$ and 0 for $x = 0$ is differentiable on all of $\mathbb{R}$ but its derivative is not continuous at 0.

6. If a function has bounded derivative on an interval then it is uniformly continuous on the interval.
   
   **T** If the derivative of a function is bounded on an interval then the function is Lipschitz (see problem 1). Lipschitz functions are uniformly continuous.

7. Every integrable function has an anti-derivative.
   
   **F** Derivatives satisfy the intermediate value property. Since not every integrable function satisfies this, not every function can be a derivative (in particular have an anti-derivative).

8. The set of integrable functions form a vector space.
   
   **T** Theorem from class.

9. The product of integrable functions is integrable.
   
   **T** Theorem from class (also easily follows from the Riemann-Lebesgue theorem).

10. The composition of integrable functions is integrable.
    
    **F** We had a counterexample to this on homework 2.