Math 6452 - Fall 2014  
Homework 2

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in only problems 4, 5, 8, 9, 12. Due: In class on September 24.

1. If $S^2$ is the unit sphere in $\mathbb{R}^3$ and $N = (0,0,1)$ and $S = (0,0,-1)$ then we have the two stereographic coordinate maps $\pi_N : (S^2 - N) \to \mathbb{R}^2$ and $\pi_S : (S^2 - S) \to \mathbb{R}^2$. If $p$ is a point in $S^2$ not equal to $N$ or $S$ then we can use the first to express a tangent vector in $T_pS^2$ in terms of the basis $\left\{ \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2} \right\}$ (where we are using Cartesian coordinates $(x^1, x^2)$ on $\mathbb{R}^2$) as

$$v = v^1 \frac{\partial}{\partial x^1} + v^2 \frac{\partial}{\partial x^2}.$$ 

Similarly we can use the second to express the same vector in terms of the basis $\left\{ \frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2} \right\}$ (where we are using Cartesian coordinates $(y^1, y^2)$ on $\mathbb{R}^2$) as

$$v = w^1 \frac{\partial}{\partial y^1} + w^2 \frac{\partial}{\partial y^2}.$$ 

Write the $w^i$ in terms of the $v^i$ (and the coordinate transform $\pi_S \circ \pi^{-1}_N$).

In particular if $\pi_N(p) = (1,0)$ and $v = \frac{\partial}{\partial x^1}$ then express $v$ in the other coordinate system.

2. Let $M$ and $N$ be two smooth manifolds.

(a) Show that for $(p,q) \in M \times N$ we have

$$T_{(p,q)}(M \times N) = (T_pM) \times (T_qN).$$

(b) If $\pi : M \times N \to M : (p,q) \mapsto p$ is the projection map then

$$df_{(p,q)} : T_{(p,q)}(M \times N) \to T_pM$$

is the projection map $(v,w) \mapsto v$.

(c) Fix a point $q_0 \in N$ and let $f : M \to M \times N : p \mapsto (p,q_0)$ then show that

$$df_p : T_pM \to T_{(p,q_0)}(M \times N)$$

is given by $v \mapsto (v,0)$.

3. Let $f : M \to N$ be a smooth map between smooth manifolds and define $F : M \to (M \times N) : p \mapsto (p, f(p))$. Show that $dF_p(v) = (v, df_p(v))$. (Here we are of course using Problem 3 (a) to write the tangent bundle of $M \times N$ as a product.)

4. If $f : M \to N$ is a submersion, then show $f$ is an open map. (That is show that for any open set $U$ in $M$ the image $f(U)$ is open in $N$.)

5. If $M$ is a compact smooth manifold and $N$ is a connected smooth manifold, then show that any smooth submersion $f : M \to N$ is surjective. Is there a submersion from $S^2$ to any $\mathbb{R}^n$, with $n > 0$?

6. Let $M$ be a compact smooth manifold and $N$ a connected smooth manifold. If they both have the same dimension and are non-empty show that any embedding $f : M \to N$ is a diffeomorphism.
7. Show that \( \mathbb{C}P^1 \) is diffeomorphic to \( S^2 \).

Hint: Using Stenographic coordinates on \( S^2 \) and our “standard” coordinates on \( \mathbb{C}P^1 \) we see both manifolds can be covered by 2 coordinate charts. Study the transition functions between these coordinate charts and see if you can define a map using the coordinate charts.

8. Define the map
\[
f : \mathbb{C}P^n \to \mathbb{C}P^m
\]
by
\[
f([x^0 : \cdots : x^n]) = [x^0 : \cdots : x^n : 0 : \cdots : 0]
\]
where \( n \leq m \). Show that \( f \) is a smooth embedding. (Notice that this says that \( S^2 \) is submanifold of \( \mathbb{C}P^2 \), or any \( \mathbb{C}P^n \) with \( n > 0 \) for that matter. Later we will see that this is a “non-trivial” \( S^2 \)).

9. With \( f \) as in the previous problem show that \( \mathbb{C}P^{n+1} - f(\mathbb{C}P^n) \) is diffeomorphic to \( \mathbb{C}P^{n+1} \).

(So for example \( \mathbb{C}P^2 \) is the union of \( \mathbb{C}P^1 \sim = S^2 \) and \( \mathbb{C}P^2 \).

Thus we can think of \( \mathbb{C}P^2 \) as the compactification of \( \mathbb{C}^2 \) by an “\( S^2 \) at infinity”.)

10. A smooth map \( f : (\mathbb{C}^{n+1} - \{(0, \ldots, 0)\}) \to (\mathbb{C}^{k+1} - \{(0, \ldots, 0)\}) \) is called homogeneous of degree \( k \) if \( f(\lambda p) = \lambda^k f(p) \) for all \( \lambda \neq 0 \) and \( p \in (\mathbb{C}^{n+1} - \{(0, \ldots, 0)\}) \). Show that \( f \) induces a map
\[
\tilde{f} : \mathbb{C}P^n \to \mathbb{C}P^k.
\]
Show this map is smooth.

11. Define the map
\[
f : \mathbb{C}P^n \times \mathbb{C}P^m \to \mathbb{C}P^{nm+n+m}
\]
by
\[
f([x^0 : \cdots : x^n], [y^0 : \cdots : y^m]) = [x^0 y^0 : x^0 y^1 : \cdots : x^0 y^m : x^1 y^0 : \cdots : x^ny^m].
\]
Show \( f \) is a smooth map and that \( f \) it is an embedding. (Notice that this shows, for example, that \( S^2 \times S^2 \) is a submanifold of \( \mathbb{C}P^3 \).)

Note: The last 4 problems could also have been carried out for real projective spaces.

12. Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a homogeneous polynomial. This implies that there is some integer \( k \) such that
\[
f(tx^1, \ldots, tx^n) = t^k f(x^1, \ldots, x^n)
\]
for all \((x^1, \ldots, x^n)\). Prove that \( f^{-1}(a) \), for \( a \neq 0 \), is an \((n-1)\)-dimensional manifold. Moreover show that if \( a \) and \( b \) are both positive then \( f^{-1}(a) \) and \( f^{-1}(b) \) are diffeomorphic and similarly if \( a \) and \( b \) are both negative. Finally show that if \( a \) and \( b \) have different signs that \( f^{-1}(a) \) and \( f^{-1}(b) \) do not have to be diffeomorphic by considering \( f(x,y,z) = x^2 + y^2 - z^2 \).

Hint: It might be good to use the famous Euler identity for homogeneous functions
\[
\sum_{i=1}^{n} x^i \frac{\partial f}{\partial x^i} = kf,
\]
(you don’t need to prove this identity, though feel free to if you like) to prove that 0 is the only critical point of \( f \). To find the diffeomorphism consider the map \( \mathbb{R}^n \to \mathbb{R}^n \) obtained by multiplication by an appropriate root of \( a/b \).