Math 6452 - Fall 2014
Homework 4

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in only problems 4, 5, 6, 8, 12. Due: In class on November 5.

1. (Problem 2 from Section 1.5 in Guillemin and Pollack) Which of the following spaces intersect transversely?
   - The $xy$-plane and the $z$-axis in $\mathbb{R}^3$.
   - The $xy$-plane and the plane spanned by $(3, 2, 0)$ and $(0, 4, -1)$ in $\mathbb{R}^3$.
   - The spaces $\mathbb{R}^k \times \{0\}$ and $\{0\} \times \mathbb{R}^l$ in $\mathbb{R}^n$. (This depends on $k, l, n$.)
   - The spaces $\mathbb{R}^k \times \{0\}$ and $\mathbb{R}^l \times \{0\}$ in $\mathbb{R}^n$. (This depends on $k, l, n$.)
   - The spaces $V \times \{0\}$ and the diagonal in $V \times V$, where $V$ is a vector space.
   - The symmetric $(A^t = A)$ and skew-symmetric $(A^t = -A)$ matrices in $M(n)$.

2. For which values of $r$ does the sphere $x^2 + y^2 + z^2 = r$ and $x^2 + y^2 - z^2 = 1$ intersect transversely? Draw the intersection for representative values of $r$.

3. A space $X$ is called contractible if the identity map is homotopic to a constant map (that is there is some point $p \in X$ such that the map $id : X \rightarrow X : x \mapsto x$ is homotopic to the map $c : X \rightarrow X : x \mapsto p$). Show that if $X$ is contractible then for any space $Y$ any two maps $Y \rightarrow X$ are homotopic. Also show that $\mathbb{R}^n$ is contractible for any $n$.

4. A space $X$ is called simply connected if every map from $S^1$ to $X$ is homotopic to a constant map. Show a contractible space is simply connected. Moreover show that the $n$-sphere $S^n$ is simply connected if $n > 1$.
   Hint: Given a smooth map $S^1 \rightarrow S^n$ use Sard’s theorem to say it misses a point and then think about stereographic projection.

5. Show that $S^n \times S^1$ is not simply connected for $n \geq 0$.
   Hint: Consider the submanifold $S = S^n \times \{p\}$ for some $p \in S^1$ and the map $f : S^1 \rightarrow S^n \times S^1 : \theta \mapsto (x, \theta)$ for some $x \in S^n$.
   Notice that problems 4 and 5 imply that $S^3$ and $S^1 \times S^2$, which are both $S^1$ bundles over $S^2$, are not diffeomorphic.

6. If $M$ and $N$ are submanifolds of $\mathbb{R}^n$ then show that for almost every $x \in \mathbb{R}^n$ the translate $M + x$ is transverse to $N$. (Here almost everywhere means “off of a set of measure zero” and $M + x = \{y + x : y \in M\}$.)

7. Suppose that $f : M \rightarrow N$ is transverse to the submanifold $S$ in $N$. Show that $T_p f^{-1} (S)$ is give by $(df_p)^{-1} (T_{f(p)} S)$. In particular if $S_1$ and $S_2$ are submanifolds of $N$ and they intersect transversely then $T_p (S_1 \cap S_2) = (T_p S_1) \cap (T_p S_2)$.

8. If $f : M \rightarrow N$ has $p$ as a regular value and $S = f^{-1} (p)$ show that the normal bundle to $S$ in $M$ is trivial.

9. Let $M$ and $N$ be manifolds of the same dimensions with $M$ compact and $N$ connected. Prove that if $f : M \rightarrow N$ has deg$_2 f \neq 0$ then $f$ is surjective.

10. Let $f : M \rightarrow \mathbb{R}$ be a smooth function. A critical point of $f$ is a point $p \in M$ such that $df_p = 0$. We say that $p$ is non-degenerate in the coordinate chart $\phi : U \rightarrow V$ if the matrix

$$H = \left( \frac{\partial^2 F}{\partial x^i \partial x^j} (q) \right)$$
is non-singular where $F = f \circ \phi^{-1}$ and $\phi(p) = q$. Show that a critical point is non-degenerate in one coordinate chart if and only if it is non-degenerate in any coordinate chart. Thus it makes sense to talk about non-degenerate critical points independent of coordinate charts.

Note: The matrix $H$ is not well-defined independent of the coordinate chart, but whether it is non-singular or not is.

11. Show that non-degenerate critical points of a function $f : M \to \mathbb{R}$ are isolated (that is each such critical point has a neighborhood containing no other critical points).

Hint: Work in local coordinate so the function is of the form $f : \mathbb{R}^k \to \mathbb{R}$ and one can then think of $df$ as a function $df : \mathbb{R}^k \to \mathbb{R}^k$. Prove $df$ is a local diffeomorphism near a non-degenerate critical point.

A function $f : M \to \mathbb{R}$ is called a *Morse function* if all of its critical points are non-degenerate.

12. Show that the function $\mathbb{R}^{n+1} \to \mathbb{R} : (x^1, \ldots, x^{n+1}) \mapsto x^{n+1}$ restricted to $S^n$ is a Morse function with exactly two critical points. (This function is sometimes called the *height function*.)

13. Suppose that $M$ is a submanifold of $\mathbb{R}^{k+1}$. The set of $v \in S^k$ for which the map $f_v : M \to \mathbb{R} : x \mapsto v \cdot x$ is not a Morse function has measure zero. (So every manifold has a lot of Morse functions.)

14. Suppose that $M$ is a submanifold of $\mathbb{R}^{k+1}$. The set of points $p \in \mathbb{R}^{k+1}$ for which the map $f_p : M \to \mathbb{R} : x \mapsto \|x - p\|^2$ is not a Morse function has measure zero.