Let $G$ be defined as all the formal symbols $x^i y^j$, $i = 0, 1$ $j = 0, \ldots n - 1$, where
\[ x^i y^j = x^{i'} y^{j'} \] if and only if $i = i'$ and $j = j'$.

\[ x^2 = e = y^n, n > 2, \]
\[ xy = y^{-1}x. \]

a. Find the form of the product $(x^i y^j)(x^k y^l) = x^a y^b$.

b. Using this, prove that $G$ is a non-abelian group of order $2n$.

c. If $n$ is odd, prove that the center of $G$ is $(e)$, while if $n$ is even the center is larger than $(e)$.

We begin by defining $x^{-1} = x$ which is consistent with $x^2 = e$. Also $x^0 = x^1 x^{-1} = x^2 = e$. Thus $ex^i = x^{0+i} = x^i = x^i e$. Likewise iterating the third equation above yields $xy^i = y^{-i}x$ or $xy^i x = y^{-i} x i = 1, \ldots n - 1$. In particular $y^n y^{-1} = y^n xy = exyx = xyx = y^{-1}xx = y^{-1} so y^{-i} = y^{-i+1}$ also follows as well as $y^i x = xy^{-i}$. Furthermore $e = y^n y = y^{-1} y = y^0$. For a. we see that with $i \in \{0, 1\}$ and $k = 0, (x^i y^j)(y^j) = x^{i+j}$. While for $i \in \{0, 1\}$ and $k = 1, (x^i y^j)(y^j) = x^{i+1} y^{j-1}$. These can be combined as $(x^i y^j)(x^k y^l) = x^{i+k} y^l x^{-1}$.

For b. we see that the above computation shows that $G$ is closed under multiplication. The above argument also shows that $e$ is the identity on $G$. For the associative law we need to show that $((x^i y^j)(x^k y^l))(x^m y^n) = (x^i y^j)(x^k y^l)(x^m y^n)$ for $i, k, n = 0$ or $1$. This can be accomplished using easily using the rules above. Thus
\[ ((x^i y^j)(x^k y^l))(x^m y^n) = (x^{i+k} y^{(l+1)n+1})(x^m y^n) = x^{i+k+n} y^{(l+1)n+1}, \]

while
\[ (x^i y^j)(x^k y^l)(x^m y^n) = (x^i y^j)(x^{n+k} y^{(l+1)n+1}) = x^{i+k+n} y^{(l+1)n+1}. \]

For the inverse we have $(x^i y^j)(y^{-j} x^{-1}) = e$ so $(x^i y^j)^{-1} = y^{j-1} x^{-1}$. The first condition on the elements of $G$ show that there are $2n$ elements. Also $x(xy) = y \neq (xy)x = y^{-1}$ since $n > 2$. For c. note that an element in the center say $x^i y^j$ must commute with $x$ and $y^j$ so $x(x^i y^j) = x^{i+1} y^j = (x^{i+1})x = x^{i+1} y^{-j}$ which by cancellation implies $y^j = y^{j-1}$ or $y^{2j} = e = y^n$. If $n$ is odd this is not possible which shows that for $n$ odd the center is $(e)$. For $n$ even the element $z = y^{n/2}$ is in the center.

Let’s write down the multiplication table for $n = 3$

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e$</td>
<td>$y$</td>
<td>$y^2$</td>
<td>$x$</td>
<td>$xy$</td>
<td>$xy^2$</td>
<td>$xy$</td>
</tr>
<tr>
<td>$y^2$</td>
<td>$e$</td>
<td>$y$</td>
<td>$xy^2$</td>
<td>$x$</td>
<td>$xy$</td>
<td>$xy^2$</td>
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<td>$x$</td>
<td>$xy$</td>
<td>$x$</td>
<td>$y$</td>
<td>$e$</td>
<td>$y$</td>
<td>$y^2$</td>
</tr>
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<td>$xy$</td>
<td>$xy^2$</td>
<td>$x$</td>
<td>$y$</td>
<td>$e$</td>
<td>$y$</td>
<td>$y^2$</td>
</tr>
<tr>
<td>$xy^2$</td>
<td>$x$</td>
<td>$xy$</td>
<td>$y$</td>
<td>$y^2$</td>
<td>$e$</td>
<td>$e$</td>
</tr>
</tbody>
</table>

The first permutation representation $x\tau_g = xg$ is read off from the columns of the above table and the second permutation is given from the rows. For example
\[ \tau_{x_2} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ x_2 & x_3 & x_1 & x_6 & x_4 & x_5 \end{pmatrix} = (1, 2, 3)(4, 6, 5) \]

\[ \tau_{x_3} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ x_3 & x_1 & x_2 & x_5 & x_6 & x_4 \end{pmatrix} = (1, 3, 2)(4, 5, 6) \]

\[ \tau_{x_4} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ x_4 & x_5 & x_6 & x_1 & x_2 & x_3 \end{pmatrix} = (1, 4)(2, 5)(3, 6) \]

\[ \tau_{x_5} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ x_5 & x_6 & x_4 & x_3 & x_1 & x_2 \end{pmatrix} = (1, 5)(2, 6)(3, 4) \]

\[ \tau_{x_6} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ x_6 & x_4 & x_5 & x_2 & x_3 & x_1 \end{pmatrix} = (1, 6)(2, 4)(3, 5) \]