Completeness

Def: \( x_n \) is a Cauchy sequence if \( \forall \varepsilon > 0 \ \exists N \text{ such that if } n, k \geq N \text{ then } d(x_n, x_k) < \varepsilon \)

Prop: \( x_n \) convergent \( \Rightarrow \) \( x_n \) is Cauchy

Proof: \( \forall \varepsilon > 0 \ \exists N \text{ such that } n, k \geq N \Rightarrow d(x_n, x_k) < \frac{\varepsilon}{2} \), where \( x = \lim_{n \to \infty} x_n \). Then, \( n, k \geq N \Rightarrow d(x_k, x_n) \leq d(x_k, x) + d(x, x_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \)

Prop: \( x_n \) Cauchy \( \Rightarrow \) \( x_n \) is also Cauchy

Proof: \( \exists M \text{ such that } n, k \geq M \Rightarrow d(x_k, x_n) < \varepsilon \).
Let \( I \) be such that \( n \geq M \). Then, if \( i, j \geq I \), \( n_i, n_j \geq n \), \( \forall i, j \geq I \),
then \( d(x_{n_i}, x_{n_j}) \leq \varepsilon \)

Prop. \( x_n \) Cauchy \( \Rightarrow \) \( x_n \) bounded

Proof. \( \varepsilon = 1 \) \( \forall N \) such that \( n, k \geq N \), \( d(x_n, x_k) < 1 \)

Let \( a = x_N \), then \( d(a, x_n) < 1 \) \( \forall n > N \)

\( R = \max \{ d(a, x_1), d(a, x_2), \ldots, d(a, x_{n-1}) \} + 1 \)

Then \( x_n \in B_R(a) \) \( \forall i \leq n \)

Prop. \( x_n \) is Cauchy, \( x_{n_i} \) is a convergent subsequence of \( x_n \),
then, \( x_n \) is convergent
Proof: Let \( a = \lim_{i \to \infty} x_n \). Let \( \epsilon > 0 \) \( \exists \) \( i \) such that \( i \geq i \)

then \( d(x_n, a) < \frac{\epsilon}{2} \). \( \exists M \) such that \( n, k \geq M \) \( d(x_n, x_k) < \frac{\epsilon}{2} \).

let \( N = \max \{ M, n_0 \} \). Let \( n \geq N \), let \( i \) be such that \( n_i \geq N \)

then \( d(x_n, a) \leq d(x_n, x_{n_0}) + d(x_{n_0}, a) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \)

Def: A metric space \( E \) is said to be complete if every Cauchy sequence in \( E \) converges in \( E \).

Prop: \( E \) complete. \( S \subseteq E \), \( S \) is closed. Then, \( S \) is complete.

Proof: \( x_n \in S \) \( x_n \) Cauchy, \( \exists a \in E \) such that \( \lim_{n \to \infty} x_n = a \)
because $E$ is complete. If $a \in S \Rightarrow \exists \varepsilon > 0 \ B_e(a) \cap S = \emptyset$.

Since $x_n \to a \in \mathbb{N}$ such that if $n \geq N$ \ $x_n \in B_e(a)$. But $x_n \in S$. Contradiction, then $a \in S$

**Theorem:** $\mathbb{R}$ is complete

**Proof:** $x_n$ Cauchy

$$S = \{x \in \mathbb{R} : x = x_n \text{ for an infinite number of } n \}$$

$x_n$ is bounded $\Rightarrow S$ is bounded from above

Let $a = \text{lub} \ S$

Let $\varepsilon > 0$. Let $N$ such that $n, k \geq N \Rightarrow |x_n - x_k| < \frac{\varepsilon}{2}$
\( a - \frac{\varepsilon}{2} \) is not an upper bound of \( S \). If \( a - \varepsilon < x \leq a \) such that \( x \in S \), then some number of \( x_n \) such that \( x_n > x \). Also \( a + \frac{\varepsilon}{2} \not\in S \), then only a finite of \( x_n \) can satisfy \( x_n \geq a + \frac{\varepsilon}{2} \).

Then an \( \infty \) of \( x_n \) satisfy \( a - \frac{\varepsilon}{2} \leq x_n \leq a + \frac{\varepsilon}{2} \).

Thus, if \( n \geq N \), and \( k \) is one of \( \uparrow \), i.e. \( |x_k - a| < \frac{\varepsilon}{2} \) and \( k \geq N \), then \( |x_n - a| \leq |x_n - x_k| + |x_k - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \).