Prop: \( S_i \) connected for all \( i \in I \). Assume \( j_0 \in I \) such that such that \( S_{j_0} \neq \emptyset \) for all \( i \in I \). Then \( \bigcup_{i \in I} S_i \) is connected.

\[ \text{Proof: } \bigcup_{i \in I} S_i \subset CA \cup B \text{, } A \text{ and } B \text{ open in } E \]

\[ AN = \emptyset. \text{ We want to show that } \bigcup_{i \in I} S_i \subset CA \text{ or } \]

\[ \bigcup_{i \in I} S_i \subset B. \text{ } S_{j_0} \text{ is connected and } S_{j_0} \subset CA \Rightarrow \]

\[ S_{j_0} \subset A \text{ or } S_{j_0} \subset B. \text{ Assume } S_{j_0} \subset CA. \text{ Let } i \in I \]
$S_i$ is connected $\Rightarrow S_i \subset A$ or $S_i \subset B$, but $S_i \cap S_i^c \neq \emptyset$

and $S_i \cap S_i^c \subset S_i^c \subset A \Rightarrow S_i \subset A$ for all $i$. Then

$\bigcup S_i \subset A$

**Th.** Intervals in $\mathbb{R}$ are connected

**Proof.** Let $I \subset \mathbb{R}$ be an interval. Assume $I \subset A \cup B$ where $A \cap B = \emptyset$ and both $A$ and $B$ are open in $\mathbb{R}$.

Assume $A \cap I \neq \emptyset$ and $B \cap I \neq \emptyset$. Let $a \in A \cap I$ and $b \in B \cap I$.
\[ A \cap (B \cap I) \]

\[ S = \{ x \in A : x \leq b \} \]

S is bounded from above. Let \( c = \text{lub} \ S \). If \( c \in A \Rightarrow \exists \varepsilon > 0 \)

such that \( B_\varepsilon(c) = (c-\varepsilon, c+\varepsilon) \subset A \) because \( A \) is open. But \( c < b \),

because \( b \in B \), then we can take \( \varepsilon < b-c \) and thus \( c+\varepsilon \in S \).

Impossible because \( c \) is an upper bound. If \( c \in B \), then \( c + \varepsilon > 0 \) and \( \varepsilon < c-a \) such that \( B_\varepsilon(c) = (c-\varepsilon, c+\varepsilon) \subset B \)

then \( c-\varepsilon \) is an other bound of \( S \) and it is smaller than \( c \),

IMPOSSIBLE. Then \( c \notin A \cup B \cap I \) impossible because \( c \in [a, b] \subset I \). Contradiction.

Then \( A \cap I = \emptyset \) or \( B \cap I = \emptyset \)
Thus \( I \) is connected.

**Continuous functions**

**Def.** \( f : E \rightarrow E' \), \( E \) and \( E' \) metric spaces. \( x_0 \in E \)

- \( f \) is continuous at \( x_0 \) if \( \forall \varepsilon > 0 \exists \delta > 0 \) such that \( d_E(x, x_0) < \delta \)

\( \Rightarrow d_E(f(x), f(x_0)) < \varepsilon \)

\[ \]

**Def.** \( f : E \rightarrow E' \), \( f \) is continuous (on \( E \)) if it is continuous...
At every $x$ in $E$

Ex: Show that $f: \mathbb{R} \rightarrow \mathbb{R}$ with $f(x) = x^2$ is continuous.

Given $\varepsilon > 0$, what is $\delta$?

$$d(f(x), f(x_0)) = |f(x) - f(x_0)| = |x^2 - x_0^2| = |x + x_0||x - x_0| \leq (|x| + |x_0|)|x - x_0| \leq (2|x_0| + 1)|x - x_0| \leq (2|x_0| + 1)\varepsilon$$

If $\varepsilon \leq 1$,

$$|x - x_0| < \delta \Rightarrow |x| \leq |x_0| + 1$$

Take $\delta = \min \{1, \frac{\varepsilon}{2|x_0| + 1}\}$
Example: \( x_0 \in E \quad f: E \rightarrow \mathbb{R} \quad f(x) = d(x, x_0) \)

Proof: \( x \in E \). Let's show \( f \) is continuous at \( x_1 \).

\[
d(f(x), f(x_1)) = |f(x) - f(x_1)| = |d(x, x_0) - d(x_1, x_0)| \leq d(x, x_1) < \delta = \epsilon
\]

Take \( \delta = \epsilon \)

\[
d(x, x_0) \leq d(x_0, x_1) + d(x_1, x) \Rightarrow d(x, x_0) - d(x_0, x_1) \leq d(x, x_1)
\]

\[
d(x_0, x_1) \leq d(x_0, x) + d(x, x_1) \Rightarrow d(x, x_0) - d(x_0, x_1) \leq d(x, x_1)
\]