Prop: $f : E \rightarrow \mathbb{R}^n \quad f(x) = (f_1(x), f_2(x), \ldots, f_n(x)) \quad x_0 \in E$

$f$ is continuous at $x_0 \iff f_i$ is continuous at $x_0$ for all $i$

Proof: $\Rightarrow$ Let $\varepsilon > 0$. \exists $\delta > 0$ such that $d(x, x_0) < \delta$ then

$d(f(x), f(x_0)) < \varepsilon$ because $f$ is continuous at $x_0$. But

$d(f(x), f(x_0)) = \sqrt{(f_1(x) - f_1(x_0))^2 + \ldots + (f_n(x) - f_n(x_0))^2}$.

$d(f(x), f(x_0)) = |f(x) - f(x_0)| \leq d(x, x_0)$ then, if $d(x, x_0) < \delta$,

$d(f(x), f(x_0)) < \varepsilon$

$\iff$ Let $\varepsilon > 0$. \exists $\delta > 0$ such that, $d(x, x_0) < \delta \Rightarrow \left| \frac{1}{n} \sum_{i=1}^{n} f_i(x) - f_i(x_0) \right| < \frac{\varepsilon}{n}$

Then $d(f(x), f(x_0)) \leq \frac{1}{n} \sum_{i=1}^{n} |f_i(x) - f_i(x_0)| < \frac{\varepsilon}{n} < \varepsilon$ as long as \( n \geq \frac{\varepsilon}{\varepsilon} \)
\[ d(x_0, x) < s = \min_{1 \leq i \leq n} S_i \]

**Theorem:** \( f : E \rightarrow E' \) is continuous, \( E \) compact, then \( f(E) \) is compact.

**Proof:** Let \( f(E) = \bigcup_{i \in I} V_i \), where \( V_i \) is open in \( f(E) \).

Then \( E = \bigcup_{i \in I} f^{-1}(V_i) \). Each \( f^{-1}(V_i) \) is open because \( f \) is continuous and each \( V_i \) is open. Then \( E = \bigcup_{i \in I} f^{-1}(V_i) \) for some \( n \) and some \( i_1, i_2, \ldots, i_n \) because \( E \) is compact. Then \( f(E) = \bigcup_{k=1}^{n} V_{i_k} \).

Then \( f(E) \) is compact.
**Def.** $f : E \rightarrow E'$. $f$ is said to be bounded if $f(E)$ is bounded.

**Corollary.** $f : E \rightarrow E'$ continuous. $E$ compact. Then $f(E)$ is compact.

Then $f$ is bounded.

**Def.** $f : E \rightarrow \mathbb{R}$. We say that $f$ attains its maximum at $x_0$ if $f(x_0) \geq f(x)$ for all $x \in E$. Similar definition for minimum.

**Corollary.** $f : E \rightarrow \mathbb{R}$ continuous. $E$ is compact, $E \neq \emptyset$. Then $f$ attains its maximum at some point $x_0$. It also attains its minimum.

**Proof.** $f(E)$ is compact. Let $a = \inf f(E)$, $a \in f(E)$. Otherwise, if $a \notin f(E)$...
\[ f(E) = \bigcup_{x \in a} \left[ f(E) \cap \{ x : x < x_i \} \right] \] and since \( f(E) \) is compact, we would have 
\[ f(E) = \bigcup_{i=1}^{n} \left[ f(E) \cap \{ x : x < x_i \} \right] \]
for some \( x_1 < x_2 < \ldots < x_n < a \), then \( x_n \) would be an upper bound on \( f(E) \) and \( x_n < a \). Contradiction.

Thus, \( a \in f(E) \), thus \( \exists x_0 \in E \) such that \( a = f(x_0) \) and thus,

since \( a = \inf_f f(E) \) we would have \( a = f(x_0) \geq f(x) \) \( \forall x \in E \)

Def. \((E,d)\) and \((E',d')\) two metric spaces. \( f : E \to E' \). We say that

\( f \) is uniformly continuous if \( \forall \varepsilon > 0 \) \( \exists \delta > 0 \) such that \( d(x,y) < \delta \)

\[ \Rightarrow d'(f(x), f(y)) < \varepsilon \]

Example: \( f(x) = x^2 \) \( f : \mathbb{R} \to \mathbb{R} \) \( f \) is continuous but
not uniformly continuous. \( \text{proof:} \)

1. Let \( x \in \mathbb{R} \). Let \( \varepsilon > 0 \).

\[
|f(x) - f(y)| = |x^2 - y^2| = |x+y||x-y| < (2|x|+1) \varepsilon \quad \text{if} \quad |x-y| < \frac{\varepsilon}{3}
\]

\[
|x+y| \leq |x| + |y| < 2|x| + 1,
\]

\[
\text{take} \quad \delta = \min \left\{ \frac{\varepsilon}{3}, \frac{1}{2|x|+1} \right\}
\]

then, \( |x-y| < \delta \implies |f(x) - f(y)| < \varepsilon \) which proves that \( f \) is continuous at \( x \).

2. Let \( \varepsilon = 1 \). Assume \( \delta \) such that \( |x-y| < \delta \implies |x^2 - y^2| < \varepsilon \)

\[
|x^2-y^2| = |x-y||x+y|
\]

\[
\text{take} \quad x = \frac{1}{\delta}, \quad y = \frac{1}{\delta} + \frac{\varepsilon}{2}
\]
\[ |x-y| = \frac{\delta}{2} \]
\[ |x+y| = \frac{2}{\delta} + \frac{\delta}{2} \]

Then
\[ |x^2-y^2| = 1 + \frac{\delta^2}{4} > \delta \]