In: \( f: E \to E' \) continuous. \( E \) compact \( \implies f \) is uniformly continuous.

Proof: Let \( \varepsilon > 0 \). \( \forall x \in E \) \( \exists \delta_x > 0 \) such that \( f(B_{\delta_x}(x)) \subseteq B_{\varepsilon/2}(f(x)) \).

\( E = \bigcup_{x \in E} B_{\delta_x}(x) \). Since \( E \) is compact, \( \exists x_1, \ldots, x_n \) such that \( E = \bigcup_{i=1}^{n} B_{\delta_{x_i}}(x_i) \).

Let \( \delta = \min_{1 \leq i \leq n} \frac{\delta_{x_i}}{2} \).

Let \( y, z \in E \) such that \( d(y, z) < \delta \).

Let \( i \) such that \( y \in B_{\delta_{x_i}/2}(x_i) \subseteq B_{\delta_{x_i}}(x_i) \).
\[ d(z, x_i) \leq d(z, y) + d(y, x) \leq \varepsilon + \frac{\varepsilon x_i}{2} \leq \frac{\varepsilon x_i}{2} + \frac{\varepsilon x_i}{2} = \varepsilon x_i \]

Thus, both \( y \) and \( z \) belong to \( B_{\frac{\varepsilon}{2}}(x_i) \), then \( d'(f(y), f(x_i)) \leq \varepsilon \) and \( d'(f(z), f(x_i)) \leq \varepsilon \). Then \( d'(f(y), f(z)) \leq d'(f(y), f(x_i)) + d'(f(x_i), f(z)) \leq \varepsilon + \frac{\varepsilon}{2} = \varepsilon \).

**Ex.** \[ f(x) = \frac{1}{x} \] : \( [0, 1] \rightarrow \mathbb{R} \) is not uniformly continuous. \([0, 1]\) is not compact.

**Ex.** \( f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \in (0, 1] \\ 0 & x = 0 \end{cases} \) : \([0, 1] \rightarrow \mathbb{R}\)
Ex. \( f(x) = \sqrt{x} \) \( f: [0,1] \to \mathbb{R} \) is uniformly continuous.

Direct proof. Let \( \varepsilon > 0 \). Let \( 0 \leq x < y < x + \delta \)

\[ \sqrt{y} - \sqrt{x} \leq \sqrt{x+\delta} - \sqrt{x} \leq \sqrt{\delta} \]

Take \( \delta = \varepsilon^2 \) \( \checkmark \)

\[ \sqrt{x+\delta} \leq \sqrt{\delta} + \sqrt{x} \] \( \checkmark \)
**Theorem**: \( f : E \to E' \) continuous. \( E \) connected \( \Rightarrow f(E) \) connected.

**Proof**: \( f \) and \( f^{-1} \) are open in \( f(E) \), \( V \cap V = \emptyset \), \( V \neq \emptyset \), \( U \neq \emptyset \).

and \( f(E) = U \cup V \), then \( E = f^{-1}(U) \cup f^{-1}(V) \) \( \Rightarrow E \) is disconnected. Contradiction.

**Corollary**: \( f : [a, b] \to \mathbb{R} \) continuous. \( \forall \lambda \) between \( f(a) \) and \( f(b) \) \( \exists c \in (a, b) \) such that \( f(c) = \lambda \).
Obs.: Balls in $\mathbb{R}^n$ are connected. Let $x$ be in the ball.

$f([0,1])$ where $f(t) = x_0 + t(x-x_0)$

is continuous. $[0,1]$ is connected, thus the segment $f_x : x_0 + t(x-x_0) \leq t \leq 1$ is also connected and $x \in f_x$, and $f_x$ is included in the ball, because $d(x_0 + t(x-x_0), x_0) = \| x_0 + t(x-x_0) - x_0 \| = t \ d(x, x_0)$

$\| z \| = \sqrt{z_1^2 + z_2^2 + \cdots + z_n^2}$ Then ball $= \bigcup f_x$. Then

the ball is the union of a collection of connected sets that
have a point in common, and thus, the ball is connected.

\[ \text{Def: } f_n : E \to E', \ n \in \mathbb{N} \]

1) \( f_n \) converges at \( x \) if \( \lim_{x \to x_0} f_n(x) \) exists.

2) \( f_n \) converges on \( E \) if \( f_n \) converges at \( x \) \( \forall x \in E \).

In this case \( f = \lim_{n \to \infty} f_n \) if \( f(x) = \lim_{n \to \infty} f_n(x) \) \( \forall x \in E \).