Q10 A) The set \( \{1, \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots \} \)

has a sup = 1 and \( \inf = \lim_{n \to \infty} \frac{1}{n} = 0 \).

B) Let \( S = \{\frac{1}{3}, \frac{4}{9}, \frac{15}{27}, \frac{40}{81}, \ldots, \frac{(3^n-1)/2}{3^n}, \ldots \} \)

thus \( \sup(S) = \lim_{n \to \infty} \frac{(3^n-1)/2}{3^n} = \lim_{n \to \infty} \frac{1}{2} \left( \frac{3^n}{3^n} \right) = \frac{1}{2} \)

and \( \inf(S) = \frac{1}{3} \).

C) Let \( S = \{\sqrt{2}, \sqrt{2+\sqrt{2}}, \ldots, a_n, \ldots \} \)

where \( a_n = \sqrt{2 + a_{n-1}} \) and \( a_0 = 0 \) \( \forall n \in \mathbb{Z}^+ \)

Suppose \( a_n \) converges to \( L \) \( \Rightarrow \lim_{n \to \infty} a_n = L \)

\( \Rightarrow L = \sqrt{2 + L} \) \( \Rightarrow L^2 = 2 + L \) \( \Rightarrow L^2 - L - 2 = 0 \) \( \Rightarrow L = 2 \text{ or } -1 \) (NA)

Thus \( \sup(S) = 2 \) and \( \inf(S) = \sqrt{2} \).

Q11

Let \( A = \{a, a^2, a^3, \ldots \} \)

Proof by contradiction.

Suppose \( A \) is bounded above; since \( A \neq \emptyset \) and \( A \subset \mathbb{R} \)

\( \Rightarrow \exists x = \sup(A) \) for \( x \in \mathbb{R} \), so \( x \leq y \) for any \( y \in \mathbb{R} \) (y is upper bound of \( A \))

we have \( a^n \leq x \) \( \forall a \in A \) thus \( a^{n+1} \leq x \) \( \Rightarrow a^n \cdot a \leq x \Rightarrow a^n \leq \frac{x}{a} \)

this implies that \( \frac{x}{a} \) is an upper bound of \( A \)

since \( a > 1 \) \( \Rightarrow \frac{x}{a} < x \) but this contradicts the statement that \( x = \text{least upper bound of } A \).

Hence, we show that \( A \) is not bounded above.
12. Since \( x \in \mathbb{R} \) and \( y \in \mathbb{R} \), and that each element of \( x \) is less than each element of \( y \), then if we pick \( x \in \mathbb{R} \), 
\( x \) either belongs to \( x \) or \( y \).

Now, since each element of \( x \) is less than each element of \( y \), \( x \) is bounded from above. Also, \( x \) is a nonempty subset of \( \mathbb{R} \), therefore, \( x \) has a L.u.b.

Let \( \alpha = \text{l.u.b.} x \). There are two cases: \( \alpha \in x \) or \( \alpha \in y \).

Assume \( \alpha \in x \), then \( \forall x \in x, x \leq \alpha \). This makes sense because then 
\( \forall y \in y, y > \alpha \) making \( \alpha \) the least upper bound of \( x \).

Hence, \( x = \{ x \in \mathbb{R} : x \leq \alpha \} \). Next, assume that \( \alpha \in y \) and \( \alpha \leq y \) for all \( y \in y \). It follows that \( \forall x \in x, x < \alpha \), since \( \alpha \in y \). Thus also makes \( \alpha \) the least upper bound of \( x \).

It follows that \( x = \{ x \in \mathbb{R} : x < \alpha \} \). Q.E.D.

14. First, suppose that \( x \) is rational. Assume \( 0 \leq a < b \), then

Let \( x = \frac{m}{n}, m, n \in \mathbb{N} \), let \( n \) be large enough so that \( \frac{1}{n} < b-a \) and choose \( m \in \mathbb{N} \) such that \( m < na \).

It follows that \( a < \frac{m}{n} \). Also, \( m \leq na \) means that

\( m-1 \leq na \).
\[ m \leq n + 1 < n(b-h)+1 = nb-1+1 = nb \]

Since \( m < nb \), then \( \frac{m}{n} < b \). It follows that \( a < \frac{m}{n} < b \).

Therefore, there exists a rational number \( x \) such that \( a < x < b \).

Now suppose \( x \) is an irrational number. By applying the above result, there exists \( y \in \mathbb{Q} \) such that \( a + \sqrt{2} < y < b + \sqrt{2} \). It follows that \( a < y - \sqrt{2} < b \). Let \( x = y - \sqrt{2} \).

Since \( y - \sqrt{2} \) is not rational, we conclude that there exists an irrational number \( x \) such that \( a < x < b \).

QED.