4. Show that the subset of $E^2$ given by $\{ (x_1, x_2) \in E^2 : x_1 > x_2 \}$ is open.

Let $E = \{ (x_1, x_2) \in E^2 : x_1 > x_2 \}$ and note that $E$ is the half-plane. It has boundary given by the line $x_2 = x_1$. Suppose that $x \in E$, so we have that $x_1 > x_2$. Then we want to determine the radius $r$ so that $B(x, r) \cap E = B(x, r) \cap \{ (x_1, x_2) \in E^2 : x_1 > x_2 \}$, which will prove that $E$ is open. Base on the distance, we choose $r = \frac{x_1 - x_2}{2r_2}$. For concreteness, take $r = \frac{x_1 - x_2}{2r_2}$. Then $B(x, r) \cap \{ (x_1, x_2) \in E^2 : x_1 > x_2 \}$ is open, and $B(x, r) \cap \{ (x_1, x_2) \in E^2 : x_1 > x_2 \} = B(x, r) \cap E$.

5. Prove that any bounded open subset of $\mathbb{R}$ is the union of disjoint open intervals.

Let $A$ denote this set and let $x \in A$. Since $A$ is open, there is an open interval containing $x$ entirely in $A$. Define $a_0 = \inf \{ y \in \mathbb{R} : (y, x) \subset A \}$ and $b_0 = \sup \{ z \in \mathbb{R} : (x, z) \subset A \}$. Since $A$ is bounded, then $a_0$ and $b_0$ are bounded. Since $A$ is open and there is an interval about point $x$ contained in $A$, so by the lub property, the sup and inf exist.

Now we note that $(a_0, b_0) \subset A$. Repeat this procedure for every point in $A$, and we have constructed a collection of open intervals whose union is $A$. Now we claim that this collection can be reduced to a disjoint collection of open intervals. Now we need to show that for any two points $x, x' \in A$, either:

- $x = x'$, and $x_0$ and $x_0$ lie in the same interval, or
- $x$ and $x'$ lie in different intervals.

Assume that $x \in (a_0, b_0)$. Suppose $x \in (a_0, b_0)$. Then since $b_0$ is the lub of all $\epsilon$ such that $(x, \epsilon) \subset A$, $b_0 \leq b_0$. Similarly, $a_0 \leq a_0$. Thus, $x \in (a_0, b_0)$.

Since $b_0$ is the lub of all $\epsilon$ such that $(x, \epsilon) \subset A$, $b_0 \leq b_0$. Similarly, $a_0 \leq a_0$. So $a_0 = a_0$ and $b_0 = b_0$.

Conversely, suppose $x \notin (a_0, b_0)$. Then $(a_0, b_0) \cap (a_0, b_0) = \emptyset$. Suppose not. Let $w$ be an element of $(a_0, b_0) \cap (a_0, b_0) = (c, d)$. By the same argument as before, $a_0 = c$, and $b_0 = d$, and also $a_0 = c$, $b_0 = d$. Hence, $a_0 = a_0$, $b_0 = b_0$, but then $x \notin (a_0, b_0)$, which is a contradiction. So $(a_0, b_0)$ and $(a_0, b_0)$ must be disjoint.
10. Prove that if \( \lim_{n \to \infty} p_n = p \) in a given metric space then the set of points \( p_1, p_2, p_3, \ldots \) is closed.

According to the Theorem on page 47, \( S \) in a metric space is closed if and only if, whenever \( p_1, p_2, p_3, \ldots \) is a sequence of points of \( S \) that is convergent in the metric space, then \( \lim_{n \to \infty} p_n \in S \).

We apply this statement with \( S = \{ p_1, p_2, p_3, \ldots \} \). We need to show that if \( q_1, q_2, \ldots \) is a sequence of points in \( S \) that is convergent, then the limit is in \( S \).

Note to the sequence \( q_1, q_2, \ldots \) we can associate a subsequence of the \( \{ p_1, p_2, p_3, \ldots \} \). Since this subsequence converges and since \( p_n \to p \) we have that \( q_n \to p \) as well. But then we have that \( q_n \in S \), and so that the set \( S \) is closed.

11. Let \( y_n = \frac{1}{n} \sum_{i=1}^{n} a_i \). Note that \( y_n - a = \sum_{i=1}^{n} \frac{a_i}{n} \leq \frac{1}{n} \sum_{i=1}^{n} |a_i| \). Since \( a_n \to L \), we can select \( N \) such that if \( n \geq N \), then \( |a_n - L| < \frac{\epsilon}{2} \). Then we have that

\[
|y_n - a| = \left| \frac{1}{n} \sum_{i=1}^{n} (a_i - a) \right| \\
\leq \frac{1}{n} \sum_{i=1}^{n} |a_i - L| \\
= R_n + \frac{1}{n} \sum_{i=n+1}^{N} |a_i - L| \\
\leq R_n + \frac{1}{n} \sum_{i=n+1}^{N} |a_i - L| < R_n + \frac{\epsilon}{2}.
\]

Now, since \( R_n \to 0 \) and \( n \to \infty \), we select \( a \geq N_1 \) such that \( n \geq N_1 \) gives \( R_n < \frac{\epsilon}{2} \). Then we have that \( |y_n - a| < \epsilon \) if \( n \geq N_1 \).

15. Let \( S^0 = \{ p \in S : p \) is an interior point of \( S \} \). We first show that \( S^0 \) is open. Let \( p \in S^0 \), so we know that \( p \) is an interior point of \( S \). Thus, there exists an open ball \( B_r(p) \subseteq S \). We claim that \( B_r(p) \subseteq S^0 \) and to see that it suffices to show that each \( g \in B_r(p) \) is an interior point of \( S \). For this \( g \in B_r(p) \), since \( B_r(p) \) is open, there is an open ball \( B_{r_1}(g) \subseteq B_r(p) \subseteq S \). Thus, \( g \) is an interior point, and \( S^0 \) is open.

We now need to prove that if \( G \subseteq S \) is an open set, then \( G \subseteq S^0 \). Let \( p \in G \), and since \( G \) is open, then there exists a ball \( B_r(p) \) such that \( B_r(p) \subseteq G \subseteq S \). This implies that \( p \) is an interior point of \( S \), and so \( p \in S^0 \).

Hence, we have \( G \subseteq S^0 \).
16. (a) $S \subseteq S^c$, and $S$ is closed if and only if $S^c = S$.

Note that from the definition we have that $S = \cap \{ F : S \subseteq F \}$.

Let $p \in S^c$ and then by De Morgan's laws, we have $S^c = \cup \{ F^c : S \subseteq F^c \}$. So there exists a closed set $F$ with $S \subseteq F^c$. Since $p \in F^c$, we have that $p \in S^c$. Thus we have that $S = S^c$, $S \subseteq S^c$.

Suppose that $S$ is closed, then $S \subseteq S^c$, and so we have that $S \subseteq S^c$ from the definition of the closure. But, by above we proved that $S \subseteq S^c$, and so $S = S^c$. Now suppose that $S = S^c$. We want to prove that $S$ is closed, and this is the same as $S^c$ being open. Since $S = S^c$, we have that $S = S^c$, and by the above, we have $S = S^c = (\cap \{ F : S \subseteq F \}) = \cup \{ F^c : S \subseteq F^c \}$.

But since each $F$ is closed, each $F^c$ is open, and since this is a union of
open sets, it is open. Thus, $S^c$ is open as desired. $S$ is closed.

(b) Let $p_n$ be a convergent sequence and $p_n \to p$. Assume $p \in S$. Since $S$ is closed, then $p \in S$. Assume $p \not\in S$, then there needs to exist a point $p_n$ such that $p_n \to p$. Let $\varepsilon > 0$, $\exists$ an integer $N$ such that $|p_n - p| < \varepsilon$. Pick an integer $N$ such that $|p_n - p| < \varepsilon$ for every $n > N$. Let $n < N$ and $p \neq p_n$, then $p_n \in B(p, \varepsilon) \cap S \implies B(p, \varepsilon) \not\subseteq S$. Let $p_n \in B(p, \varepsilon) \cap S$ and $S = \frac{1}{n}$. Then, $|p_n - p| < \frac{1}{n} \implies p_n \not\in S$.

(c) This is the same as saying that $S = \{ p \in E : d(p, S) = 0 \}$. For this to be true, then $\forall \varepsilon > 0$, $B(p, \varepsilon) \cap S \neq \emptyset$ and $B(p, \varepsilon) \cap S^c = \emptyset$. $B(p, \varepsilon) \cap S^c \neq \emptyset$ is true when $S = \frac{1}{n}$ and $p_n \in B(p, n) \cap S \implies p \in S$.

for $B(p, \varepsilon) \cap S^c \neq \emptyset$, let $p$ be an interior point of $S$. By definition of an interior point, $B(p, \varepsilon) \subseteq S$ which contradicts $B(p, \varepsilon) \cap S^c \neq \emptyset$. Therefore, $p \not\in S$. So $B(p, \varepsilon) \cap S^c \neq \emptyset$. So $B(p, \varepsilon)$ contains at least one point of $S$. So, a point $p \in E$ is in $S$ iff a ball in $E$ of center $p$ contains points of $S$. If $p$ is not an interior point of $S^c$.  

Total 30