Problem 62. Let $X$ be a topological space. Show that $X$ path connected implies $X$ connected.

Proof. Suppose that $X$ is path connected but not connected. Then there exist disjoint, nonempty open sets $G_1, G_2$ such that $X = G_1 \cup G_2$. Since $G_1 \neq \emptyset \neq G_2$, we know that there is $x \in G_1$ and $y \in G_2$. Also, the sets are disjoint, so $x \neq y$. Since $X$ is path connected, there exists a path $f : [0,1] \to X$ such that $f(0) = x$ and $f(1) = y$. Since $f$ is continuous, we have that $f^{-1}(G_1), f^{-1}(G_2)$ are open in $[0,1]$. Furthermore, since $G_1 \cap G_2 = \emptyset$, $f^{-1}(G_1) \cap f^{-1}(G_2)$ is also empty. But since $X = G_1 \cup G_2$, we must have that $[0,1] = f^{-1}(G_1) \cup f^{-1}(G_2)$, so we then have that $[0,1]$ is not connected, which is a contradiction. □

Problem 65. Show that no two of the spaces $(0,1)$, $(0,1]$, and $[0,1]$ are homeomorphic.

Proof. First note that $[0,1]$ is compact, since it is a closed, bounded subspace of $\mathbb{R}$. Also, $(0,1)$ and $(0,1]$ are not compact, since they are not closed. Thus, since we know that if $X$ is compact and $f$ is continuous, then $f(X)$ is compact, we have that $[0,1]$ cannot be homeomorphic to $(0,1]$ or $(0,1)$, as the image of $[0,1]$ under a homeomorphism would have to be the whole space, which is not compact.

To establish that $(0,1)$ and $(0,1]$ are not homeomorphic, suppose that there were a homeomorphism $f : (0,1) \to (0,1]$. Then let $a := f^{-1}(1)$. There is such a unique $a$ since $f$ is a bijection. Now consider the restriction $g$ of $f$ to $(0,a) \cup (a,1)$. The function $g$ is a homeomorphism of $(0,a) \cup (a,1)$ with $(0,1)$, since it is a restriction of a continuous function and $g^{-1}$ is also a restriction of a continuous function. (Bijectivity is obvious since we’ve removed one point from the domain and its corresponding point from the image.) But we cannot have that $(0,a) \cup (a,1)$ and $(0,1)$ are homeomorphic, since the first space is disconnected and the second is connected, contrary to the fact that homeomorphisms preserve connectivity. □

Problem 67. Let $S^1 := \{ x \in \mathbb{R}^2 \mid \|x\|_2 = 1 \}$. Let $f : S^1 \to \mathbb{R}$ be continuous. Show that there exists $x \in S^1$ such that $f(x) = f(-x)$.

Proof. Suppose that there is no such $x$. Then we have that $f(x) \neq f(-x)$ for all $x \in S^1$. Now consider the function

$$g(x) := \frac{f(x) - f(-x)}{|f(x) - f(-x)|}$$

Since we never have $f(x) = f(-x)$, the denominator is never 0, so $g$ is well defined. It is also continuous by the continuity of $f$. Observe that $g(x) = \pm 1$ for all $x \in S^1$. 

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In fact, since $g$ is continuous and $S^1$ is connected, we must have that $g(S^1)$ is connected, and thus either $g(x) = 1$ for all $x \in S^1$ or $g(x) = -1$ for all $x \in S^1$. Thus, we either have $f(x) > f(-x)$ for all $x$ or $f(x) < f(-x)$ for all $x$. But then we have

$$f(-x) = f(x) > f(-x) > f(-x) = f(x),$$

which is a contradiction. Thus there must be an $x \in S^1$ such that $f(x) = f(-x)$. □

**Problem 69.** True or false (justify):

(a) $X, Y$ path connected implies $X \times Y$ path connected

(b) $A \subseteq X$, $A$ path connected implies $\overline{A}$ path connected

(c) $X$ path connected and $f: X \rightarrow Y$ implies $f(X)$ path connected

(d) $A_\alpha \subseteq X$ path connected for all $\alpha \in \Lambda$ and $\cap_{\alpha \in \Lambda} A_\alpha \neq \{\}$ implies that $\bigcup_{\alpha \in \Lambda} A_\alpha$ is path connected.

**Proof.**

(a) This statement is true. Let $(x_1, y_1)$ and $(x_2, y_2)$ be points of $X \times Y$. Since $X$ and $Y$ are path connected, there are paths $f_1: [0, 1] \rightarrow X$ and $f_2: [0, 1] \rightarrow Y$ such that $f_1(0) = x_1, f_1(1) = x_2, f_2(0) = y_1$, and $f_2(1) = y_2$. We define a new function $g: [0, 1] \rightarrow X \times Y$ by $g(s) = (f_1(s), f_2(s))$. Then we have $g(0) = (x_1, y_1)$ and $g(1) = (x_2, y_2)$ as desired. Also, the image of $g$ is contained in $X \times Y$ by construction. Thus, it only remains to show that $g$ is continuous. Fix $s_0 \in [0, 1]$ and let $G$ be any neighborhood of $g(s_0)$. Then $G = \cup_{\alpha \in \Lambda} U_\alpha \times V_\alpha$ for some sets $U_\alpha \subseteq X$, $V_\alpha \subseteq Y$ and index set $\Lambda$. We need to find a neighborhood $U$ of $s_0$ such that $g(U) \subseteq G$. To do this, fix any $\alpha_0 \in \Lambda$ such that $g(s_0) \in U_{\alpha_0} \times V_{\alpha_0}$. Now consider $f_1^{-1}(U_{\alpha_0})$ and $f_2^{-1}(V_{\alpha_0})$. Since $f_1$ and $f_2$ are continuous, these sets are open and contain $s_0$. Thus, $f_1^{-1}(U_{\alpha_0}) \cap f_2^{-1}(V_{\alpha_0})$ is an open subset of $[0, 1]$ containing $s_0$. Furthermore, we must have $g(f_1^{-1}(U_{\alpha_0}) \cap f_2^{-1}(V_{\alpha_0})) \subseteq U_{\alpha_0} \times V_{\alpha_0} \subseteq G$, and thus $g$ is continuous.

(b) This statement is false. Consider the set $A = \{(x, \sin(1/x)) \mid x \in (0, 1]\}$. We clearly have that

$$\overline{A} = \{(x, \sin(1/x)) \mid x \in (0, 1]\} \cup \{(0) \times [-1, 1]\}.$$  

We certainly have that $A$ is path connected. However, $\overline{A}$ is not path connected, as there is no way to get from a point in $\{0\} \times [-1, 1]$ to a point of $A$, as $\sin(1/x)$ oscillates infinitely many times as we approach $x = 0$.

(c) This statement is false, since there is no continuity assumption on $f$. Let $X, Y = [0, 1]$ and $f(0) = 0$ and $f(x) = 1$ for all $x \in (0, 1]$. Then $f(X) = \{0, 1\}$ which is not path connected as a subspace of $[0, 1]$.

(d) This statement is true. Let $x, y \in \cup_{\alpha \in \Lambda} A_\alpha$. Then there are $\alpha_1, \alpha_2 \in \Lambda$ such that $x \in A_{\alpha_1}$ and $y \in A_{\alpha_2}$. If $\alpha_1 = \alpha_2$, we’re done since $A_{\alpha_1}$ is path connected. Otherwise, we use the fact that $\cap_{\alpha \in \Lambda} A_\alpha \neq \{\}$, which gives that there is $x_0 \in A_{\alpha_1} \cap A_{\alpha_2}$. Since $A_{\alpha_1}$ is path connected, there is a path $f_1: [0, 1] \rightarrow A_{\alpha_1}$ such that $f_1(0) = x$ and $f_1(1) = x_0$. Similarly, there is a path $f_2: [0, 1] \rightarrow A_{\alpha_2}$ such that $f_2(0) = x_0$ and $f_2(1) = y$. Thus we construct a path $f: [0, 1] \rightarrow \cup_{\alpha \in \Lambda} A_\alpha$ by defining

$$f(x) := \begin{cases} f_1(2x) & \text{if } x \in [0, 1/2] \\ f_2(2x - 1) & \text{if } x \in [1/2, 0]. \end{cases}$$
The function $f$ is clearly continuous by construction, and thus the union of the $A_\alpha$ is path connected as desired. □