Modeling the dynamics of asset prices

Discrete modeling

$[0, T] \quad S_t = T/N$

$S_t = S$ at time $t \leq T$

(1) $S_{t+1} = u_t S_t$

(2) $u_t$ are non-negative random variables, $S_0$ given

(3) $u_t$ are i.i.d.

Set $Z_t = \log u_t$

(4) $\log S_{t+1} = \log S_t + Z_t$
(5) \( z_t \sim N(\mu_t, \sigma^2_t) \)

Thus \( u_t \) is lognormally distributed.

**Obs:**

a) \( \log S_t = \log S_{t-1} + \varepsilon_{t-1} = \log S_{t-2} + \varepsilon_{t-2} + \varepsilon_{t-1} \)

\( \log S_t = \log S_0 + \varepsilon_0 + \varepsilon_1 + \cdots + \varepsilon_{t-1} \)

b) \( \log S_t \sim N(\log S_0 + \sqrt{t} \mu_t, \sigma^2 S_t \varepsilon) \)

\( L_S(t) = t \quad S_t = S(t) \)

\( \log S(t) \sim N(\log S_0 + \sqrt{t}, \sigma^2 t) \)

Continuous time modeling

\( dt = \delta t \quad t = L_S t \)
(6) \( \log S(t+dt) - \log S(t) = \zeta \) 

Set \( \zeta = \nu dt + \sigma (W(t+dt) - W(t)) \) \((7)\) 

Then \( \sigma (W(t+dt) - W(t)) = \zeta - \nu dt \sim N(0, \sigma^2 dt) \)

**Def.** \( X(t) \) is a stochastic process if \( X(t) \) is a random variable for each \( t \).

**Notation** \( \delta X(t) = X(t+dt) - X(t) \)

(8) \( \delta \log S(t) = \nu dt + \sigma \delta W(t) \)

**Notation.** \( \int_s^t \delta X(r) = \sum_{i=0}^{N-1} \delta X(s+i \cdot dt) = \sum_{i=0}^{N-1} X(s+i \cdot dt) - X(s+i \cdot dt) \)

\( = X(t) - X(s) \) where \( dt = \frac{t-s}{N} \)
\[
\begin{align*}
\text{Obs. : } \sigma (W(t) - W(s)) &= \sum_{i=0}^{N-1} \sigma \, dW(s+idt) = \\
\sum_{i=0}^{N-1} \sigma (W(s+(i+1)dt) - W(s+idt)) &\sim N(0, \sigma^2 Nd dt) = N(0, \sigma^2 (t-s))
\end{align*}
\]

**Def.:** \( W(t) \) stochastic process is a standard Wiener process if

1) \( W(0) = 0 \)

2) \( W(t) - W(s) \sim N(0, t-s) \)

3) \( t_1 < t_2 < t_3 < t_4 \) then \( W(t_2) - W(t_1) \) and \( W(t_4) - W(t_3) \) are independent

**Obs.:** 3) comes from the \( \mathcal{Z}_t \) being independent

**Stochastic integrals**
Def. $W(t)$ Wiener process, $X(t)$ stochastic process that depends on $W(s)$ for $0 \leq s < t$.

Set $t_k = \frac{kT}{N}$, $t_0 = 0$, $t_1$, $T = t_N$.

$$\int_0^T X(t) \, dW(t) = \lim_{N \to \infty} \sum_{k=0}^{N-1} X(t_k) \left[ W(t_{k+1}) - W(t_k) \right]$$

(Obs: $X(t_k)$ and $W(t_{k+1}) - W(t_k)$ are independent)

Thus

$$E \left[ \int_0^T X(t) \, dW(t) \right] \approx E \left[ \sum_{k=0}^{N-1} X(t_k) \left[ W(t_{k+1}) - W(t_k) \right] \right] =$$

$$= \sum_{k=0}^{N-1} E[X(t_k)] E[W(t_{k+1}) - W(t_k)] = 0$$
Ito's lemma. \( W = \) Wiener process

\[(a) \quad dX = a(X, t) \, dt + b(X, t) \, dW \]

\( G = G(x, y) \) a function of two variables

Taylor expansion

\[ dG = \frac{\partial G}{\partial x} \, dx + \frac{\partial G}{\partial y} \, dy + \frac{1}{2} \left( \frac{\partial^2 G}{\partial x^2} \right) (dx)^2 + \frac{1}{2} \left( \frac{\partial^2 G}{\partial y^2} \right) (dy)^2 \]

\[ + \ldots \]

\[ F = F(X, t) \], where \( X \) satisfies \((a)\)

Note: \( dW = W(t+dt) - W(t) \sim N(0, dt) \)

Thus \( dW = \Phi(t) \sqrt{dt} \)

where \( \Phi(t) \sim N(0, 1) \)

Back to \( F \)
\[ dF = \partial F \frac{dx}{dt} + \partial F \frac{dt}{dt} + \frac{1}{2} \partial^2 F \frac{(dx)^2}{dt^2} + O((dt)^{3/2}) \]

\[ dF = \partial F \frac{dx}{dt} \left( a dt + b dW \right) + \partial F \frac{dt}{dt} + \frac{1}{2} \partial^2 F b^2 (dW)^2 + O((dt)^{3/2}) \]

\[ (dW)^2 \rightarrow dt \]

**Ito's lemma:** Wiener

\[ dX = a(X,t) dt + b(X,t) dW \]

If \( F = F(X,t) \), then

\[ dF = \left( a \frac{\partial F}{\partial x} + b \frac{\partial F}{\partial t} + \frac{1}{2} b^2 \frac{\partial^2 F}{\partial x^2} \right) dt + b \frac{\partial F}{\partial x} dW \]
Geometric Brownian motion is defined by the stochastic differential equation

\[ dS(t) = \mu S(t) \, dt + \sigma S(t) \, dW(t) \]

Wiener

\[ X = S, \quad \alpha = \mu S, \quad b = \sigma S \]

\( \mu \) and \( \sigma \) are constants \( \mu = \text{drift}, \quad \sigma = \text{volatility} \)

Let \( F(S,t) = \log S \)

\[ \frac{\partial F}{\partial S} = \frac{1}{S}, \quad \frac{\partial F}{\partial t} = 0, \quad \frac{\partial^2 F}{\partial S^2} = -\frac{1}{S^2} \]

\[ dF = \left( \mu S \frac{1}{S} - \frac{1}{2} \sigma^2 S^2 \frac{1}{S^2} \right) dt + \sigma S \frac{1}{S} \, dW \]
\[ d \log S = (\mu - \frac{1}{2} \sigma^2) \, dt + \sigma \, dW \]

Integrate:

\[ \log S(t) = \log S(0) + (\mu - \frac{1}{2} \sigma^2) \, t + \sigma \, W(t) \]

\[ \log S(t) \sim \mathcal{N} \left[ \log S(0) + (\mu - \frac{1}{2} \sigma^2) \, t, \sigma^2 \, t \right] \]

\[ S(t) = S(0) \, e^{(\mu - \frac{\sigma^2}{2}) \, t + \sigma W(t)} \]

\( S(t) \) is lognormally distributed

\[ E[S(t)] = S(0) \, e^{\mu t} \]

Application of the no-arbitrage principle
1) Price of a forward contract

A forward contract is an agreement to sell an asset at a specified future date at a predetermined price $F$. 

$r =$ risk free interest rate

Portfolio 1 (person selling the asset at time $T$ for $F$)

a) Borrow $S(0)$ at time 0 to buy the asset
b) Sell for $F$ at time $T$
c) Pay back the loan, i.e. pay back $S(0)e^{rT}$

If $F > S(0)e^{rT}$, then he made a risk free profit from zero investment. Not allowed by the no arbitrage principle.
Thus \( F \leq S(0) e^{rT} \)

Under a portfolio (person buying the asset at time \( T \) for \( F \))

a) Sell short the asset at time 0 to get \( S(0) \)
b) Invest that money with the risk free interest to have \( S(0) e^{rT} \) at time \( T \)
c) Buy the asset at time \( T \) for \( F \)

If \( S(0) e^{rT} - F > 0 \), then he made a risk free profit with 0 initial investment. Not allowed by the no arbitrage principle. Thus

\[
F \geq S(0) e^{rT}
\]

Thus \( F = S(0) e^{rT} \)
Put-call parity (No arbitrage principle. Example)

C and P are the prices of a call and a put, both European, with the same exercise price \( K \) and maturity \( T \).

**Portfolio \( P_1 \) at \( t=0 \)**
- one call option + \( Ke^{-RT} \) in cash

**Portfolio \( P_2 \) at \( t=0 \)**
- one put option + 1 share of the underlying

**Value of \( P_1 \) at \( t=0 \)**  
\( C + Ke^{-RT} \)

**Value of \( P_2 \) at \( t=0 \)**  
\( P + S(0) \)
Value of $P_1$ at $t=T$ = \[ \begin{cases} \frac{s(t) - k + ke^{-2T}e^{2T}}{2} & \text{if } s(t) > k \\ ke^{-2T}e^{2T} & \text{if } s(t) \leq k \end{cases} \]

Value of $P_2$ at $t=T$ = $\max\left(\frac{s(t)}{2}, k\right)$

Value of $P_2$ at $t=T$ = $\max\left(\frac{s(t)}{2}, k\right)$

Value of $P_1$ at $t=T$ = $\max\left(\frac{s(t)}{2}, k\right)$

Thus, no arbitrage principle implies
Value of $P_1$ at time 0 = Value of $P_2$ at time 0

\[ C + K e^{-rT} = P + S(0) \]