Black-Scholes model

\[ dS = \mu S \, dt + \sigma S \, dW \]

\( f(S,t) \) = value of the option at time \( t \) when the value of the asset at that time is \( S \)

Portfolio: Short in an option + long in \( A \) shares of stock

Value of portfolio = \( g = A S - f(S,t) \)

Reminder of Itô's lemma

\[ dX = a \, dt + b \, dW \quad \text{and} \quad F(X,t) \text{ then} \]

\[ dF = \left( a \frac{\partial F}{\partial X} + b \frac{\partial F}{\partial t} + \frac{1}{2} b^2 \frac{\partial^2 F}{\partial X^2} \right) \, dt + b \frac{\partial F}{\partial X} \, dW \]
In our case \( X = S \), \( a = \mu S \), \( b = \sigma S \) and \( F = g \)

\[
dq = (\mu S \frac{\partial g}{\partial S} + \frac{\partial g}{\partial t} + \frac{1}{2} (\sigma S)^2 \frac{\partial^2 g}{\partial S^2}) \, dt + \sigma S \frac{\partial g}{\partial S} \, dW
\]

\[
dq = \left[ \mu S \left( A - \frac{\partial f}{\partial S} \right) - \frac{\partial f}{\partial t} + \frac{1}{2} (\sigma S)^2 \left( -\frac{\partial^2 f}{\partial S^2} \right) \right] \, dt + \sigma S \left[ A - \frac{\partial f}{\partial S} \right] \, dW
\]

set \( A = \frac{\partial f}{\partial S} (S, t) \) in \( \left[ t, t+dt \right] \)

\[
dq = (-\frac{\partial f}{\partial t} - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}) \, dt
\]

So in \( dt \), \( g \) increases by \((-\frac{\partial f}{\partial t} - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}) \, dt \) no randomness

Thus, no arbitrage implies that this should be equal to
\[ \frac{rg}{r} \, dt = - \left( s \frac{\partial f}{\partial s} - f \right) \, dt \]

Then,

\[ \frac{\partial f}{\partial t} + S \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} - rf = 0 \]

Black-Scholes equations

Assumptions used to get Black-Scholes

1) No arbitrage

2) Frictionless market (no transaction costs, borrowing and lending interest rates are the same, etc....)
3) Asset price follows a geometric Brownian motion
4) r & σ are constants for 0 ≤ t ≤ T. No dividends are paid in this period of time. The option is European

Need to impose terminal conditions to solve Black-Scholes, i.e. we need to prescribe \( f(S,T) \), i.e. \( f \) at \( t = T \). This is the payoff function

For calls \( f(S,T) = \max(S - K, 0) \)

For puts \( f(S,T) = \max(K - S, 0) \)
Binomial

\[ S_0 = \text{asset price at } t=0 \]
\[ S_t = \text{asset price at } t = \Delta t \]
\[ t=0 \quad t=\Delta t \]

\[ \begin{aligned}
  &uS_0 \quad p(S_t = uS_0) = p_u \quad u > d \\
  &dS_0 \quad p(S_t = dS_0) = p_d \\
  &\quad \quad p_u + p_d = 1 \quad p_u > 0 \quad \& \quad p_d > 0
\end{aligned} \]

European option

Exercise time = \Delta t

\[ f_0 = \text{value of the option at } t=0 \]
Goal: find $f_0$

$f_u =$ value of option at $t = s T$ if $S_t = u S_0$

$f_d =$ value of option at $t = s T$ if $S_t = d S_0$

Obs: 1) We know $f_u$ & $f_d$

If it is a call option

$$f_u = \max \left\{ u S_0 - K, 0 \right\}$$

$$f_d = \max \left\{ d S_0 - K, 0 \right\}$$

If it is a put option

$$f_u = \max \left\{ K - u S_0, 0 \right\}$$

$$f_d = \max \left\{ K - d S_0, 0 \right\}$$

2) No arbitrage $\Rightarrow \quad d < e^{r s T} < u$
Portfolio \( P_1 \): an option

Portfolio \( P_2 \): \( \alpha \) shares of the asset + \( \beta \) cash

At \( t = sT \)

<table>
<thead>
<tr>
<th>( S_t )</th>
<th>Value of ( P_2 )</th>
<th>Value of ( P_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s_{0u} )</td>
<td>( \alpha s_{0u} + \beta e^{r sT} )</td>
<td>( f_u )</td>
</tr>
<tr>
<td>( s_{0d} )</td>
<td>( \alpha s_{0d} + \beta e^{-r sT} )</td>
<td>( f_d )</td>
</tr>
</tbody>
</table>

Select \( \alpha \) \& \( \beta \) so value of \( P_1 = \) value of \( P_2 \) at \( t = sT \)

Regardless of the value of \( S_t \)
\[ x S_0 u + \beta e^{rst} = f_u \]
\[ x S_0 d + \beta e^{rst} = f_d \]

Then

\[ x = \frac{f_u - f_d}{(u-d)S_0} \]
\[ \beta = \frac{(uf_d - uf_u)}{d-u} \]

Since value of \( P_1 \) = value of \( P_2 \) at \( t = \infty \), no arbitrage implies value of \( P_1 \) = value of \( P_2 \) at \( t = 0 \)

Value of \( P_2 \) at \( t = 0 \) = \( x S_0 + \beta \)

Value of \( P_1 \) at \( t = 0 \) = \( f_0 \)
\[ f_0 = \alpha S_0 + \beta = \frac{f_u - f_d}{u-d} + \frac{(u-f_d)(u-d)}{u-d} e^{-rst} \]

\[ f_0 = e^{-rst} \left\{ \frac{e^{rst} - d}{u-d} f_u + \frac{u - e^{rst}}{u-d} f_d \right\} \]

**Def.**
- \( T_u = \frac{e^{rst} - d}{u-d} \)
- \( T_d = \frac{u - e^{rst}}{u-d} \)

**Obs.**
1. \( T_u + T_d = 1 \)
2. \( T_u > 0 \) \& \( T_d > 0 \). So we can interpret them as probability.

**Def.**
- \( f_i \): value of the option at \( t = st \)

**Obs.**
- \( P(f_i = f_u) = P(S_i = uS_0) = p_u \). Note that \( f_0 \) is independent of \( p_u \).
\[ E(f_i) = p_u f_u + p_d f_d \]

**Definition:** \( E(f_i) = \pi_u f_u + \pi_d f_d \) is the expected value with respect to this new probability \( \pi_u \) and \( \pi_d \).

**Notation:** This "artificial probability" \( \pi_u \) & \( \pi_d \) is called risk-neutral.

\[ \text{As } f_0 = e^{-\text{rst}} E(f_i) \]

\[ \text{As } p_u = \pi_u \quad \pi_d = \pi_d \quad \iff \quad E(S_i) = \hat{E}(S_i) \]

**Algorithm (One step binomial)**
Input: $S_0, u, d, K, r, \Delta t$

Output: $f_0$

If call then

$$f_u = \max \left\{ S_0 u - K, 0 \right\}$$
$$f_d = \max \left\{ S_0 d - K, 0 \right\}$$

If put then

$$f_u = \max \left\{ K - S_0 u, 0 \right\}$$
$$f_d = \max \left\{ K - S_0 d, 0 \right\}$$

$$\Pi_u = \frac{e^{r \Delta t} - d}{u - d}$$
\[ T_{d} = 1 - T_{u} \]

\[ f_{0} = e^{-rS} \left( T_{u} f_{u} + T_{d} f_{d} \right) \]

Calibration: How do we pick \( u, d \) and \( p \)?

We assume the asset value follows a geometric Brownian motion

\[ ds = \mu S \, dt + \sigma S \, dW \]

We are interested in the value of the option, which does not depend on \( \mu \). If we change \( \mu \), the value of the option does not change. Note

\[ E(S) = S(0) e^{\mu t} \]
After $S_t$, $E(S(st)) = S(0) e^{\mu st}$

If we change $\mu$ by $\gamma$, we will get the risk-neutral probabilities. This is a widely adopted strategy that we will follow:

$$dS = \gamma S dt + \sigma S dW$$

$$\log(S(st)) \sim N(\log(S(0)) + (\gamma - \frac{\sigma^2}{2}) st, \sigma^2 st)$$

Thus,

$$E[S(st)] = S(0) e^{\gamma st}$$  \hspace{1cm} (1)$$

$$Var[S(st)] = (S(0))^2 e^{2\gamma st} (e^{\sigma^2 st} - 1)$$  \hspace{1cm} (2)$$

Pick $\mu, d$ & $p_{mu}$ so the expectation and variance using the
binomial are the same as equations (1) and (2)

Set \( u = \frac{1}{d} \) (using the freedom we have because we have two equations with 3 unknowns)

From binomial: \( p = p_\infty \quad S = S_0 = S(0) \)

\[
E[S(st)] = E[S_i] = pSu + (1-p)Sd \quad (3)
\]

\[
\text{Var}[S(st)] = E[S_i^2] - (E[S_i])^2 = p(Su)^2 + (1-p)(Sd)^2 - (pSu + (1-p)Sd)^2 \quad (4)
\]