Reminder

Richardson extrapolation

\[ T(h) = a_0 + a_1 h^{p_1} + a_2 h^{p_2} + \ldots \]

\[ 0 < p_1 < p_2 < \ldots \]

\[ A_{00} = T(h) \]

\[ A_{10} = T(h/2) \rightarrow A_{01} = A_{10} + \frac{A_{10} - A_{00}}{2^{p_1}} \]

\[ A_{20} = T(h/4) \rightarrow A_{11} = A_{20} + \frac{(A_{20} - A_{10})}{2^{p_1-1}} \rightarrow A_{02} = A_{11} + \frac{(A_{11} - A_{01})}{2^{p_2-1}} \]

\[ A_{30} = T(h/8) \rightarrow A_{21} = A_{30} + \frac{(A_{30} - A_{20})}{2^{p_1-1}} \rightarrow A_{12} = A_{21} + \frac{(A_{21} - A_{11})}{2^{p_2-1}} \]
Fact: Let $y(x, h)$ be the solution of $y' = f(x, y)$ and $y(x_0) = y_0$ with Euler's method with step $h$. Let $y(x)$ be the exact solution. Then

$$y(x, h) = y(x) + c_1(x) h + c_2(x) h^2 + \ldots$$

$p_1 = 1, \quad p_2 = 2, \quad p_3 = 3, \ldots$

We can use Richardson extrapolation:

Passive extrapolation

\[ x_0 \quad x_1 \quad x_2 \quad x_n \]
Get $y(x_n, h)$ and then extrapolate

$y(x_n, h/2)$

Active extrapolation

Get $y(x_i, h)$ extrapolate to get a better approximation $y(x_i, h/2)$ of $y(x_i)$. Call this approximation $y_i$.

We do $n$ extrapolations instead of 1.
Note. Varying steps.

First order vector valued odes

\[ y = y(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_n(x) \end{bmatrix} \]

\[ F = F(x, y) = \begin{bmatrix} f_1(x, y_1, \ldots, y_n) \\ f_2(x, y_1, \ldots, y_n) \\ \vdots \\ f_n(x, y_1, \ldots, y_n) \end{bmatrix} \]

\[ y_1' = F(x, y) \quad \text{or} \quad y_i' = f_i(x, y_1, \ldots, y_n) \quad \text{I.e.} \quad y(x_0) = y_0 \]

\[ y_2' = f_2(x, y_1, \ldots, y_n) \]
\[ y_n' = f_n(x, y_1, \ldots, y_n) \]

Example

\[ y_1' = 3y_1 + y_2 \]
\[ y_2' = y_1 - y_2 \]

\[ \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 3y_1 + y_2 \\ y_1 - y_2 \end{bmatrix} \]

Obs. Euler, Improved Euler and Runge-Kutta can all be used to solve vector valued odes.

Second order odes

\[ y'' = f(x, y, y') \]
Set $z_1 = y$ and $z_2 = y'$ then $\bigcirc$ is equivalent to

$z_1' = z_2$

$z_2' = f(x, z_1, z_2)$

We transformed the second order equation to a system of two first order equations

$z_1' = F(x, z)$

$F = \begin{bmatrix}
  z_2 \\
  f(x, z_1, z_2)
\end{bmatrix}$

If the initial condition was $y(x_0) = y_0$ and $y'(x_0) = y_1$, now it is $z(x_0) = \begin{bmatrix}
  z_1(x_0) \\
  z_2(x_0)
\end{bmatrix} = \begin{bmatrix}
  y_0 \\
  y_1
\end{bmatrix}$.
**Boundary value problems**

\[ y'' = f(x, y, y') \quad y(a) = A \quad \text{and} \quad y(b) = B \]

Find \( y(x) \) for \( a \leq x \leq b \)

**Auxiliary problem**

\[ y(x, s) \quad y' = \frac{dy}{dx} \]

Let \( \phi(s) = y(b, s) - B \)

Find \( s^* \) zero of \( \phi \)

Then \( y(x, s^*) \) is the solution to the boundary value problem

\[ y'' = f(x, y, y') \]

\( y(a) = A \)

\( y'(a) = s \)
Two options to find the zero of $\phi(s)$

1) Bisection. Need to be able to compute $\phi(s)$.

2) Newton-Raphson. $s_{n+1} = s_n - \frac{\phi(s_n)}{\phi'(s_n)}$

Need to be able to evaluate $\phi'(s)$

$y'' = f(x, y, y')$ $y(x_0) = A$ and $y'(x_0) = s$

Change to system

\[
\begin{cases}
   z_1' = z_2 \\
   z_2' = f(x, z_1, z_2) \\
   z_1(x_0) = A \\
   z_2(x_0) = s
\end{cases}
\]
Take derivatives of $\mathbf{z}$ with respect to $s$

\[
\begin{align*}
\left( \frac{\partial \mathbf{z}_1}{\partial s} \right)' &= \frac{\partial^2 \mathbf{z}_2}{\partial s^2} \\
\left( \frac{\partial \mathbf{z}_2}{\partial s} \right)' &= \frac{\partial f(x, \mathbf{z}_1, \mathbf{z}_2)}{\partial \mathbf{z}_1} \frac{\partial \mathbf{z}_1}{\partial s} + \frac{\partial f(x, \mathbf{z}_1, \mathbf{z}_2)}{\partial \mathbf{z}_2} \frac{\partial \mathbf{z}_2}{\partial s} \\
\frac{\partial \mathbf{z}_2}{\partial s} (a) &= 0 \\
\frac{\partial \mathbf{z}_2}{\partial s} (a) &= 1
\end{align*}
\]

Let $\mathbf{Z}_3 = \frac{\partial \mathbf{z}_1}{\partial s}$ and $\mathbf{Z}_4 = \frac{\partial \mathbf{z}_2}{\partial s}$

\[
\begin{align*}
\mathbf{z}_1 &= \mathbf{z}_2 \\
\mathbf{z}_2 &= f(x, \mathbf{z}_1, \mathbf{z}_2)
\end{align*}
\]
\[
\begin{aligned}
\begin{cases}
z_3 = z_4 \\
z_4 = \frac{\partial}{\partial z_1} (x, z_1, z_2) \ z_3 + \frac{\partial}{\partial z_2} (x, z_1, z_2) \ z_4
\end{cases}
\end{aligned}
\]

\[
\begin{aligned}
&z_1(a) = A \\
&z_2(a) = s \\
&z_3(a) = 0 \\
&z_4(a) = 1
\end{aligned}
\]

We solve Eqs. and \text{JC} all 4 simultaneously.

\[
\begin{aligned}
\phi'(s) &= z_1(b) - B \\
\phi'(s) &= \frac{\partial}{\partial s} (b, s) = 2 z_1(b) = z_2(b)
\end{aligned}
\]
$\phi_{s}(s) = z_{3}(b)$

$s_{n+1} = s_{n} - \frac{(z_{1}(b) - 13)}{z_{3}(b)}$