Local analysis

Local behavior of solutions

Local means near a point

Reminder: A function \( f(x) \) is analytic at \( x_0 \) if

\[
f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n \quad \text{for all } x \text{ such that } |x-x_0| < \epsilon
\]

for some \( \epsilon > 0 \). In this case

\[
a_n = \frac{f^{(n)}(x_0)}{n!}
\]

Eq (1) \( y^{(n)} + p_{n-1}(x) y^{(n-1)} + \ldots + p_0(x) y = 0 \)

Ordinary points: \( x_0 \) is an ordinary point of

Eq (1) if \( p_0(x), p_1(x), \ldots, p_{n-1}(x) \) are analytic at \( x = x_0 \)

Example (1) \( y'' = e^x y \)

\( y'' - e^x y = 0 \quad p_0 = -e^x \quad p_1 = 0 \)

all the points are ordinary points in this equation.

(2) \( y'' - \frac{y'}{x} = 0 \quad x = 0 \) is not an ordinary point

because \( p_0 = -\frac{1}{x} \) is not analytic at \( x = 0 \).

Eq (2) \( y'' + p_1(x) y' + p_0(x) y = 0 \)

Theorem. Let \( x_0 \) be an ordinary point of
Eq.(2). then there are two linearly independent solutions of Eq.(2) that are analytic at \( x = x_0 \).

Example (1) \( y' = 2xy \) and \( y(0) = 1 \). Find the power series of \( y \) near \( x = 0 \).

\[
y = \sum_{n=0}^{\infty} a_n x^n
\]

\[
y' = \sum_{n=1}^{\infty} n a_n x^{n-1}
\]

Plug into equation

\[
\sum_{n=1}^{\infty} n a_n x^{n-1} = 2x \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} 2a_n x^{n+1}
\]

\[
\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} 2a_n x^{n+1} = 0
\]

\[
\sum_{k=0}^{\infty} (k+1) a_{k+1} x^k = \sum_{k=1}^{\infty} 2a_{k-1} x^k = 0
\]

\[
a_1 + \sum_{k=1}^{\infty} [(k+1) a_{k+1} - 2a_{k-1}] x^k = 0
\]

\[
a_1 = 0
\]

\[(k+1) a_{k+1} - 2a_{k-1} = 0 \quad n = k-1\]
\[(n+2) a_{n+2} - 2a_n = 0\]

\[a_{n+2} = \frac{2}{n+2} a_n\]

\[y = \sum_{n=0}^{\infty} a_n x^n\]

\[y(0) = a_0 = 1\]

\[a_0 = 1\]
\[a_1 = 0\]
\[a_2 = 1\]
\[a_3 = 0\]
\[a_4 = \frac{1}{2}\]
\[a_6 = \frac{1}{3} \cdot \frac{1}{2}\]

\[a_8 = \frac{1}{4} \cdot \frac{1}{3} \cdot \frac{1}{2}\]
\[a_{10} = \frac{1}{5} \cdot \frac{1}{4} \cdot \frac{1}{3} \cdot \frac{1}{2}\]

\[y(x) = \sum_{n=0}^{\infty} a_{2n} x^{2n} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = e^{x^2}\]

**Check**

\[y' = 2x y\]

\[y' = 2x e^{x^2} = 2x y\]  \(\checkmark\)

A second order example \(y'' = x y\)  \(\text{(3)}\)

Find two linearly independent solutions of \(y'' = x y\)

\[y = \sum_{n=0}^{\infty} a_n x^n\]

\[y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\]

Plug into \(\text{eq (3)}\)

\[\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+1}\]

\[\sum_{n=0}^{\infty} a_n x^{n-2} = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+1}\]
\[ \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} a_n x^{n+1} \geq a_n x \]

\[ \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k = \sum_{\Gamma=1}^{\infty} a_{\Gamma-1} x^{\Gamma} \]

\[ \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k - \sum_{k=1}^{\infty} a_{k-1} x^k = 0 \]

\[ 2a_2 + \sum_{k=1}^{\infty} \left[ (k+2)(k+1) a_{k+2} - a_{k-1} \right] x^k = 0 \]

\[ a_2 = 0 \]

\[ (k+2)(k+1) a_{k+2} - a_{k-1} = 0 \]

\[ a_{k+2} = \frac{a_{k-1}}{(k+2)(k+1)} \quad (k \geq 1) \]

- \[ a_0 \text{ anything} \]
- \[ a_1 \text{ anything} \]
- \[ a_2 = 0 \]

\[ a_3 = \frac{a_0}{3.2} \quad k=4 \quad a_6 = \frac{a_3}{6.5} \frac{a_6}{65.32} \]

Set \[ a_0 = 1, a_1 = 0 \]

\[ a_1 = \frac{x^3}{3.2} + \frac{x_6}{65.32} + \frac{x^9}{9.8, 65.32} + \cdots \]

Set \[ a_0 = 0, a_1 = 1 \]
Set $a_0 = 0$, $a_1 = 1$

\[ y_2 = x + \frac{x^4}{4.3} + \frac{x^7}{7.6.4.3} + \frac{x^{10}}{10.9.7.6.4.3} + \ldots \]

**Regular singular points**

\[ y'' + p(x) y' + q(x) y = 0 \]

$x_0$ is a regular singular point if

\[(x-x_0) p(x) \text{ and } (x-x_0)^2 p(x) \text{ are both analytic}\]

**Example (1)**

\[ y'' + \frac{y'}{x} = 0 \]

$p_1 = 0$, $p_0 = \frac{1}{x}$ then $xp_1 = 0$ and $x^2 p_0 = x$ are both analytic

so $x_0 = 0$ is a regular singular point.

**Theorem:** If $x_0$ is a regular singular point, there is one solution of the form

\[ y = (x-x_0)^\alpha \sum_{n=0}^{\infty} a_n (x-x_0)^n, \quad a_0 \neq 0 \]

Radius of convergence $> 0$

**Example:**

\[ y' - \frac{y}{x} = 0 \]

Find a solution of the form (4) when $x_0 = 0$

$x(-1) = -1$ is both analytic.
\[ x(-1) = -1 \] is both analytic with \( a_0 \neq 0 \)

Set \[ y = (x-x_0)^\alpha \sum_{n=0}^{\infty} a_n (x-x_0)^n \]

\[ y = \sum_{n=0}^{\infty} a_n (x-x_0)^{n+\alpha} \]

\[ y' = \sum_{n=0}^{\infty} a_n (n+\alpha) (x-x_0)^{n+\alpha-1} \]

Plug into eq

\[ \sum_{n=0}^{\infty} a_n (n+\alpha) x^{n+\alpha-1} - \sum_{n=0}^{\infty} a_n x^{n+\alpha-1} = 0 \]

\[ \sum_{n=0}^{\infty} a_n (n+\alpha-1) x^{n+\alpha-1} = 0 \]

We need \( a_n (n+\alpha-1) = 0 \) for all \( n \geq 0 \)

\[ a_0 (\alpha-1) = 0 \quad \text{\( \Rightarrow \) Then} \quad \alpha = 1 \]

\[ a_1 \alpha = 0 \quad \Rightarrow \quad a_1 = 0 \]

\[ a_2 (\alpha+1) = 0 \quad \Rightarrow \quad a_2 = 0 \]

\( \alpha = 1 \) \( a_0 \neq 0 \)

\[ y = a_0 x \]

Check \[ y' - \frac{y}{x} = 0 \]

\[ a_0 - a_0 = 0 \quad \checkmark \]