Sets in the complex plane

$|z-z_0| = r$ circle of radius $r$ centered at $z_0$

$|z-z_0| < r$ open disk of radius $r$ centered at $z_0$

Open sets: A set $D$ in the complex plane is said to be open if it does not contain any of the points in its boundary.

Example: All open disks are open sets.

Closed sets: A set $D$ is closed if it contains all its boundary points.

Example 1) $|z-z_0| \leq 3$ is closed

2) The whole complex plane $\mathbb{C}$ is both closed and open
3) D is neither closed nor open.

**Def.** A neighborhood of a point \( z_0 \) is an open disk centered at \( z_0 \). (We did section 17.3)

**Functions of a complex variable**

The domain and the range are both sets of complex numbers.

**Example** \( f(z) = z + 2z^2 \)

\[
f(1+i) = 1+i + 2(1+i)^2 = 1+i + 2(1+2i+i^2) = 1+i + 2(2i) = 1+5i
\]

**Obs.** \( z = x+iy \)

\[
f(z) = u(x,y) + iv(x,y)
\]

\( u \) and \( v \) are real valued functions, \( x \) and \( y \) are real numbers

**Example** \( f(z) = z + 2z^2 \) \( z = x+iy \)
\[ f(z) = x + iy + 2(x + iy)^2 = x + iy + 2(x^2 + 2ixy - y^2) = \]
\[ = (x + 2x^2 - 2y^2) + i(y + 4xy) \]
\[ f(z) = u(x, y) + i \, v(x, y) \]
\[ u(x, y) = x + 2x^2 - 2y^2 \quad v(x, y) = y + 4xy \]

**Obs.** \( \lim_{z \to z_0} f(z) = L \)

**Example** \( \lim_{z \to (1+i)} z + 2z^2 = (1+i) + 2(1+i)^2 = 1 + 5i \)

\[ \lim_{z \to i} \frac{1}{z - i} = \text{does not exist} \]

\[ \lim_{z \to i} \frac{z - i}{z^2 + 1} = \lim_{z \to i} \frac{(z - i)}{(z - i)(z + i)} = \lim_{z \to i} \frac{1}{z + i} = \frac{1}{2i} = \frac{-i}{2} \]

**Continuity.** \( f \) is continuous at \( z_0 \) if \[ \lim_{z \to z_0} f(z) = f(z_0) \]

**Obs.** All polynomials are continuous. Rational functions are continuous except at the zeros of the denominator.

**Derivatives.** Let \( f \) be a function defined in \( \mathbb{C} \). The derivative of \( f \)
a neighborhood of \( z_0 \). The derivative of \( f \) at \( z_0 \) is

\[
f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}
\]

**Def:** We say that \( f \) is differentiable at \( z_0 \) if \( f'(z_0) \) exists.

**Example** \( f(z) = z^2 \)

\[
f'(z_0 + h) - f(z_0) = (z_0 + h)^2 - z_0^2 = \frac{(z_0 + h)^2 - z_0^2}{h} = 2z_0h + h^2
\]

\[
= \frac{2z_0h + h^2}{h} = 2z_0 + h
\]

\[
\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = \lim_{h \to 0} 2z_0 + h = 2z_0
\]

\[
f'(z) = \frac{d}{dz}(z^2) = \frac{df(z)}{dz} = 2z
\]

**Rules:**

1. \( C \) constant \( \frac{d}{dz} C = 0 \)

2. \( \frac{d}{dz} \left[ f(z) + g(z) \right] = f'(z) + g'(z) \)

3. \( \frac{d}{dz} [f(z)g(z)] = f'(z)g(z) + f(z)g'(z) \)
3) $\frac{d}{dz}(f(z)g(z)) = f(z)g(z) + f(z)g'(z)$

4) $\frac{d}{dz}\left(\frac{f(z)}{g(z)}\right) = \frac{f(z)g(z) - f(z)g'(z)}{(g(z))^2}$

5) $\frac{d}{dz}f(g(z)) = f'(g(z))g'(z)$

6) $\frac{d}{dz}z^n = nz^{n-1}$

**Example**

\[ f(z) = \text{Re}(z) \quad f(z) = x \]

\[ h = h_1 + ih_2 \quad z = x + iy \]

\[ \frac{f(z+h) - f(z)}{h} = \frac{x+h_1 - x}{h_1 + ih_2} = \frac{h_1}{h_1 + ih_2} \]

\[ h = 0 + ih_2 \]

\[ h = h_1 + 0 \]

\[ \lim_{h_1 \to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h_1 \to 0} \frac{h_1}{h_1 + ih_2} = 1 \]

\[ \lim_{h_2 \to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h_2 \to 0} \frac{0}{h_2 + 0 + ih_2} = 0 \]

\[ h_1 = 0 \]
This means that $f(z)$ does not exist.

**Def:** A complex valued function is said to be analytic at $z_0$ if it is differentiable at any point in a neighborhood of $z_0$. (We did section 17.4)

**Cauchy-Riemann equations**

$$f(z) = u(x, y) + iv(x, y) \quad h = h_1 + ih_2$$

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

$$f'(z) = \lim_{h_1 \to 0, h_2 = 0} \frac{f(z+h) - f(z)}{h} = \lim_{h_1 \to 0} \frac{u(x+h_1, y) + iv(x+h_1, y) - u(x, y) - iv(x, y)}{h_1}$$

$$= \lim_{h_1 \to 0} \frac{u(x+h_1, y) - u(x, y)}{h_1} + \lim_{h_1 \to 0} \frac{iv(x+h_1, y) - iv(x, y)}{h_1}$$

$$= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

$$f'(z) = \lim_{h_2 \to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h_2 \to 0} \frac{u(x, y+h_2) + iv(x, y+h_2) - u(x, y) - iv(x, y)}{h_2}$$

$$= \lim_{h_2 \to 0} \frac{u(x, y+h_2) - u(x, y)}{h_2}$$
\[ \lim_{h_2 \to 0} \frac{u(x,y+h_2) - u(x,y)}{h_2} = \frac{\partial u}{\partial y} \]

then

\[ \frac{\partial u}{\partial x} + i \frac{\partial u}{\partial y} = \frac{\partial \overline{u}}{\partial x} - i \frac{\partial \overline{u}}{\partial y} \]

these are called the Cauchy–Riemann equations.

**Theorem:** \( f(x) = u(x,y) + i(v(x,y)) \). 

\( f \) is differentiable at \( z_0 \) if and only if the Cauchy–Riemann equations are satisfied at \( z_0 \).

**Example:** 1) \( f(z) = x \)

\[ u = x \quad \frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = 0 \]

\( v = 0 \)

then \( f \) is not differentiable.

2) \( f(z) = z + 2z^2 \)
\[ f(z) = x + iy + 2(x + iy)^2 = (x + 2x^2 - 2y^2) + i(y + 4xy) \]

\[ u = x + 2x^2 - 2y^2 \]

\[ v = y + 4xy \]

\[ \frac{\partial u}{\partial x} = 1 + 4x \quad \Rightarrow \quad \frac{\partial v}{\partial y} = 1 + 4x \]

\[ \frac{\partial u}{\partial x} = 4y \quad \Rightarrow \quad -\frac{\partial u}{\partial y} = 4y \]

The Cauchy–Riemann equations are satisfied.

Thus \( f(z) = z + 2z^2 \) is differentiable everywhere.

Obs: \( f \) is analytic. Then

\[ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \]

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial y} \right) = \]

\[ = \frac{\partial}{\partial x} \left( \frac{\partial v}{\partial y} \right) + \frac{\partial}{\partial y} \left( -\frac{\partial v}{\partial x} \right) = \]

\[ = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0 \]
\[ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \] is called the Laplace's equation.

\underline{Obs.:} the real and the imaginary parts of analytic functions satisfy Laplace's equation.

\underline{Obs.:} the converse is also true.

Any function that satisfies Laplace's equation is the real part and the imaginary part of some analytic functions.