\[ e^z = e^{x+iy} = e^x (\cos y + i \sin y) \]

Observations:
1. \[ e^{z_1 + z_2} = e^{z_1} e^{z_2} \]
2. \[ e^{z_1 - z_2} = \frac{e^{z_1}}{e^{z_2}} \]
3. \[ f = u + iv \]

\[ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{CREE} \]

\[ f = e^{x+iy} = e^x \cos y + i e^x \sin y \]
\[ u = e^x \cos y \quad v = e^x \sin y \]

\[ \frac{\partial u}{\partial x} = e^x \cos y \quad \frac{\partial u}{\partial y} = e^x \cos y \quad \checkmark \]

\[ \frac{\partial v}{\partial y} = -e^x \sin y \quad -\frac{\partial v}{\partial x} = -e^x \sin y \quad \checkmark \]

The CREE are satisfied when \( f(z) = e^z \)

Thus, \( e^z \) is analytic everywhere.

\[ f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial u}{\partial y} = e^x \cos y + i e^x \sin y = e^z \]

\[ f(z) = e^z \quad \text{for all } z \in \mathbb{C} \]
\[
\frac{d}{dz} e^z = e^z \quad \text{for all } z \in \mathbb{C}
\]

Logarithms

\[ w = \ln z \quad \text{if} \quad e^w = z \]

Need to find \( w \) as a function of \( z \)

\[ w = u + iv \]
\[ z = x + iy \]

\[ e^w = e^u \cos \varphi + i e^u \sin \varphi = x + iy \]

\[ e^u \cos \varphi = x \quad \text{and} \quad e^u \sin \varphi = y \]

\[ e^u = \sqrt{x^2 + y^2} = |z| \quad \text{if} \quad u = \ln |z| \]

\[ \tan \varphi = \frac{y}{x} \]

\[ z = e^u (\cos \varphi + i \sin \varphi) \]

\[ \theta = \arg(z) \]
\[ v = \theta + 2\pi k \quad k \text{ an integer} \]
\[ \ln z = w = u + i v = \ln |z| + i (\arg(z) + 2\pi k) \]

\[ \ln z = \ln |z| + i (\arg(z) + 2\pi k) \quad k \text{ an integer} \]

Not a problem, \( e^z \) is not one to one

\[ e^0 = 1 \]

\[ e^{2\pi i} = \cos(2\pi) + i \sin(2\pi) = 1 \]

\[ e^{2\pi i} = e^0 \]

\[ e^z = e^z + 2\pi k i \quad k \text{ integer} \]

**Trigonometric functions** \( y \in \mathbb{R} \)

\[ e^{iy} = \cos y + i \sin y \]

\[ e^{-iy} = \cos y - i \sin y \]

\[ \cos y = \frac{e^{iy} + e^{-iy}}{2} \quad \text{and} \quad \sin y = \frac{e^{iy} - e^{-iy}}{2i} \]

For complex numbers, we define

\[ \cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \text{and} \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i} \]

The trigonometric identities are also valid for complex numbers.

\[ \frac{d}{dz} \cos z = -\sin z \]

\[ \frac{d}{dz} \sin z = \cos z \]
\[
\frac{\alpha \cos z}{dz} = -\sin z \\
\frac{\alpha \sin z}{dz} = \cos z
\]

We did section 17.7

**Integrals**

\[\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f(z_k^*) \cdot (z_k - z_{k-1}) = \int f(z) \, dz\]

**Evaluation of integrals**

**Parametrize \( \gamma \)**

\[\gamma(t) \quad a \leq t \leq b\]

\[\int_{\gamma} f(z) \, dz = \int_{a}^{b} f(\gamma(t)) \cdot \gamma'(t) \, dt\]

**Example** \( \gamma \) is given by \( x = 3t \), \( y = t^2 \), \(-1 \leq t \leq 4\)
Example

Given

\[ -1 \leq t \leq 4 \]

Evaluate \( \int_{\gamma} \frac{1}{z} \, dz \)

\[ \int_{-1}^{4} \frac{x(t) + iy(t)}{x(t) + iy(t)} (x(t) + iy(t)) \, dt = \]

\[ = \int_{-1}^{4} (3t - i t^2) (3 + 2i t) \, dt = \int_{-1}^{4} (9t + 2t^3) \, dt + \]

\[ + i \int_{-1}^{4} (-3t^2 + 6t^2) \, dt = \frac{9}{2} (16 - 1) + \frac{1}{2} (4^4 - 1) + \]

\[ + i (4^3 + 1) \]

Example

\[ \int_{\gamma} \frac{1}{z} \, dz \]

\( \gamma \) is the circle centered at 0 of radius \( r \) in the counter clockwise direction

\[ z(t) = re^{it}, \quad 0 \leq t \leq 2\pi \]

\[ \int_{\gamma} \frac{1}{z} \, dz = \int_{0}^{2\pi} \frac{1}{z(t)} \, dt = \int_{0}^{2\pi} i re^{it} \, dt = \]
\[ \oint_{\gamma} \frac{1}{z} \, dz = \int_{0}^{2\pi} \frac{1}{z(t)} \, dt = \int_{0}^{2\pi} \frac{ire^{it}}{re^{it}} \, dt = \int_{0}^{2\pi} ire^{it} \, dt = i \int_{0}^{2\pi} dt = 2\pi i \]

Properties of integrals

1. \[ \int_{\gamma} cf(z) \, dz = c \int_{\gamma} f(z) \, dz \]

2. \[ \int_{\gamma} [f(z) + g(z)] \, dz = \int_{\gamma} f(z) \, dz + \int_{\gamma} g(z) \, dz \]

3. \[ \int_{\gamma} f(z) \, dz = \int_{\gamma_1} f(z) \, dz + \int_{\gamma_2} f(z) \, dz \]

4. \[ \int_{\gamma_1} f(z) \, dz = -\int_{\gamma_2} f(z) \, dz \]
-$\mathcal{C}$ is the curve $\mathcal{C}$ but with opposite orientation.

We did 18.1

Cauchy–Goursat theorem

**Def:** A set $D$ is simply connected if it is connected and it does not have holes.

**Example**

This set is not connected.

It is connected but not simply connected.

It is **NOT** simply connected. It has a hole, the hole is a single point.

**Theorem:** Let $D$ be simply connected.

Let $f$ be analytic in $D$. Then, if $\mathcal{C}$ is
A closed curve inside \( D \), then
\[
\oint_{\Gamma} f(z) \, dz = 0
\]

Example
\[ f(z) = \frac{1}{z} \]
\[ \Gamma = \text{circle of radius 1} \]
\[
\oint_{\Gamma} \frac{1}{z} \, dz = 2\pi i
\]
\[
\Gamma \quad \text{is not analytic at 0, so there is no contradiction}
\]
\[ \frac{1}{z} \]

\[ D \quad \text{circle of radius one centered at 2} \]
\[
\oint_{\Gamma} \frac{1}{z} \, dz = 0 \quad \text{because} \quad \frac{1}{z}
\]
\[
\Gamma \quad \text{is analytic inside} \quad \Gamma
\]
\[ D \quad \text{is the red set} \]
\[ D \quad \text{is not simply connected} \]
\[ f \quad \text{is analytic in} \quad D \]
If $f$ is analytic in $D$, then
\[ \int_{C_1} f(z) \, dz = \int_{C_2} f(z) \, dz \]
if $f$ is analytic in the region inside $C_1$ but also outside $C_2$.

Why?

\[ \int_{C_1} f(z) \, dz + \int_{C_3} f(z) \, dz + \int_{C_4} f(z) \, dz = 0 \]

\[ \int_{C_3} f(z) \, dz + \int_{C_4} f(z) \, dz = 0 \]

\[ \int_{C_1} f(z) \, dz + \int_{C_2} f(z) \, dz \]

\[ \int_{C_1} f(z) \, dz = \int_{C_2} f(z) \, dz \]