Power series

\[ \sum_{k=0}^{\infty} a_k z^k = \lim_{N \to \infty} \sum_{k=0}^{N} a_k z^k \]

There exists \( R > 0 \) (\( R \) could be \( \infty \)) such that
the limit exists for all \( |z| < R \) but the
limit does not exist for all \( |z| > R \).
This \( R \) is called the radius of convergence.

Facts:
1) Let \( f(z) = \sum_{k=0}^{\infty} a_k z^k \). \( f(z) \) is analytic for \( |z| < R \).

Moreover,
\[ a_k = \frac{f^{(k)}(0)}{k!} \]

2) \( f'(z) = \sum_{k=1}^{\infty} k a_k z^{k-1} \). The radius of convergence is also \( R \).

3) \( F(z) = \sum_{k=0}^{\infty} \frac{a_k}{k+1} z^{k+1} \). This series also has \( R \)
as its radius of convergence and \( F(z) = f(z) \).

2) \( f(z) \) be analytic in \( D \)
\( z_0 \in D \). Assume that \( z \in D \) whenever \( |z-z_0| < R \). Then,
\[ \left|\frac{(k)}{z-z_0}\right| < \frac{(k)}{R} \] for all \( |z-z_0| < R \).
\[
f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z-z_0)^k \quad \text{for all } |z-z_0| < r
\]

This is the Taylor series.

Example 1) \( e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} \)

2) \( f(z) = \frac{1}{1-z} = \sum_{k=0}^{\infty} z^k \), this is true for all \( |z| < 1 \)

Laurent series

Let \( f \) be analytic in \( D \).

Assume \( R < |z-z_0| < r \) is included in \( D \).

Then \( f \) can be expanded as a Laurent series, i.e.

\[
f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k \quad \text{for all } r < |z-z_0| < R
\]

Example 1) \( f(z) = \frac{1}{z} \)
2) \[ f(z) = \frac{1}{z(1-z)} = \frac{1}{z} \left(1 + z + z^2 + \cdots\right) = \sum_{k=0}^{\infty} z^k \]

\[
eq \frac{1}{z} + 1 + z + z^2 + \cdots = \sum_{k=-1}^{\infty} z^k
\]

This is a Laurent series.

Fact:

\[
\oint_{\gamma} \frac{1}{z-z_0} \, dz = \left\{ \begin{array}{ll}
\pi i & k \neq -1 \\
0 & k = -1
\end{array} \right.
\]

Def: \[ f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k \]

The residue of \( f \) at \( z_0 \) is \( a_{-1} \).

Example \[ f(z) = \frac{1}{(1-z)^2} = \frac{1}{z} + 1 + z + z^2 + \cdots \]

\[ \text{Res}(f, 0) = 1 \]

\[ \text{Res}(f, 1) = ? \]
\[
\frac{1}{z(z-1)} = \frac{(-1)}{(z-1)} \frac{1}{1-[(-1)(z-1)]} \leq \sum_{k=0}^{\infty} \frac{(-1)^k}{(z-1)^k} = \frac{(-1)}{(z-1)} \sum_{k} \left(1 - (z-1) + (z-1)^2 \ldots \right) = \\
\frac{(-1)}{(z-1)} + 1 - (z-1) \ldots \ldots \ldots \ldots \\
\text{Res} \left( \frac{1}{z(z-1)}, 1 \right) = -1
\]

\text{Obs.} \quad \text{Assume } f \text{ is analytic in a domain } D \text{ except at some points } z_1, z_2, \ldots, z_n. \text{ We call these points singularities.}

\text{Theorem:} \quad \text{Assume } z_1, z_2, \ldots, z_n \text{ are the only singularities of } f \text{ inside the curve } \gamma \text{ then}
\[
\oint_{\gamma} f(z) \, dz = 2\pi i \sum_{k=1}^{n} \text{Res}(f, z_k)
\]
\[ f(z) = \sum_{k=1}^{\infty} \]

(Observe: Assume \( f(z) = \frac{g(z)}{(z-z_0)} \) where \( g \) is analytic around \( z_0 \) and \( g(z_0) \neq 0 \). Then \( \text{Res}(f, z_0) = g(z_0) \)

**Example** \( f(z) = \frac{1}{1+z^2} \)

\[
\text{Res}(\frac{1}{1+z^2}, i) = g(i) = \frac{1}{2i} = \frac{-i}{2}
\]

\[
f(z) = \frac{1}{(z-i)(z+i)} = \frac{g(z)}{(z-i)}
\]

where \( g(z) = \frac{1}{z+i} \)

**Example** Compute \( \oint_{\gamma} \frac{1}{1+z^2} \, dz \)

\( \gamma \) is the circle of radius 7 centered at 0

\[
\text{Res}(\frac{1}{1+z^2}, i) = \frac{-i}{2}
\]

\[
\frac{1}{\gamma} = \frac{1}{\gamma \cap (z+i)} = \frac{1}{\gamma \cap (z-i)}
\]
\[ \frac{1}{1 + z^2} = \frac{1}{(z-i)(z+i)} = \frac{1}{2z} \left( \frac{1}{z-i} - \frac{1}{z+i} \right) \]

\[ \text{Res} \left( \frac{1}{1 + z^2}, -i \right) = 0(-i) = \frac{1}{2i} = \frac{i}{2} \]

\[ \oint \frac{dz}{1 + z^2} = 2\pi i \left\{ \text{Res} \left( \frac{1}{1 + z^2}, i \right) + \text{Res} \left( \frac{1}{1 + z^2}, -i \right) \right\} = 2\pi i \left\{ -\frac{i}{2} + \frac{i}{2} \right\} = 0 \]

We did most of Chapter 19.