Obs: \( f \) is analytic inside a closed curve \( \Gamma \), then \( \oint_{\Gamma} f = 0 \) if analytic in the red region.

Obs: \( \ell_1 \) and \( \ell_2 \) have the same initial and final points \((\ell_1 \cup \ell_2) \subset D \) open simply connected set. \( f \) is analytic in \( D \). Then

\[
\oint_{\ell_1} f(z) \, dz = \oint_{\ell_2} f(z) \, dz
\]
Theorem: $f$ is analytic in $D$, open simply connected $\mathcal{C} \subset D$. Then $\int_{\mathcal{C}} f(z) \, dz$ depends only on the initial and end points $A$ of $\mathcal{C}$.

Example: $f(z) = \frac{1}{z}$

$\mathcal{C}_1 = \{ |z| = 1, \text{Im}(z) > 0 \}$

Parametrization of $\mathcal{C}_1$: $z(t) = e^{it}$, $0 \leq t \leq \pi$

$\mathcal{C}_2 = \{ |z| = 1, \text{Im}(z) < 0 \}$

Parametrization of $\mathcal{C}_2$: $z(t) = e^{-it}$, $0 \leq t \leq \pi$

\[
\int_{\mathcal{C}_1} \frac{dz}{z} = \int_0^\pi \frac{i e^{it}}{e^{it}} \, dt = i\pi \]

$D$ is not simply connected

\[
\int_{\mathcal{C}_2} \frac{dz}{z} = \int_0^\pi \frac{-i e^{-it}}{e^{-it}} \, dt = -i\pi \]
Theorem: $D$ simply connected, $f$ analytic in $D$

If $f$ inside $D$, $z_0$ and $z$, the initial and final points of $f$. $F'(z) = f(z)$ for all $z$ in $D$. Then

\[ \int_{\gamma} f(z) \, dz = F(z_1) - F(z_0) \]

Example $\quad f(z) = z^2$

\[ \int_{\gamma_1} z^2 \, dz = \int_{0}^{1} t^2 \, dt + \int_{1}^{i} (t+1)^2 \, dt = \frac{1}{3} - \frac{i}{3} \]

$\gamma = \gamma_1 \cup \gamma_2$
\[ F(z) = \frac{z^3}{3} \quad F'(z) = f(z) \quad D = C \quad \text{then} \]

\[ \int_C z^2 \, dz = F(1) - F(-1) = \frac{i^3}{3} - \frac{(-i)^3}{3} = \frac{1-i}{3} \]

Theorem: \( D \) simply connected, \( f \) analytic in \( D \). Then there exists \( F \) in \( D \) such that \( F'(z) = f(z) \) for all \( z \in D \).

Cauchy's Integral Formula

\( D \) simply connected. \( \gamma \) closed curve. \( \gamma \subseteq D \). \( \gamma \) does not intersect itself. \( z_0 \) is a point inside \( \gamma \). then
\[ f(z_0) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z-z_0} \, dz \]

**Proof:**

\[ \oint_{\gamma} \frac{f(z)}{z-z_0} \, dz = \oint_{\gamma} \frac{f(z)}{z-z_0} \, dz = \oint_{\gamma} \frac{f(z)}{z-z_0} \, dz = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z-z_0} \, dz = \oint_{\gamma} f(z) \, dz = 2\pi i f(z_0) \]

\[ \lim_{\varepsilon \to 0} \oint_{\gamma_{\varepsilon}} f(z) \, dz = 0 \]
Example: \[ \oint_{|z| = 2} \frac{z^2 - 4z + 4}{z + i} \, dz \]

\[ |z| = 2 \quad z_0 = -i \quad f(z) = z^2 - 4z + 4 \]

\[ f(z_0) = \frac{1}{2\pi i} \oint_{|z| = 2} \frac{f(z)}{z - z_0} \, dz \]

\[ \oint_{|z| = 2} \frac{z^2 - 4z + 4}{z + i} \, dz = 2\pi i f(-i) = 2\pi i (-i)^2 - 4(-i) + 4) = 2\pi i (3 + 4i) \]

\[ |z| = 2 \]
Power series

\[ \sum_{k=0}^{\infty} a_k z^k = \lim_{N \to \infty} \sum_{k=0}^{N} a_k z^k \]

There exists \( R > 0 \) (\( R \) could be \( \infty \)) such that the limit exists for all \( |z| < R \), but the limit does not exist for all \( |z| > R \). \( R \) is called the radius of convergence.

**Facts:**
1. \( f(z) = \sum_{k=0}^{\infty} a_k z^k \). \( f(z) \) is analytic for \( |z| < R \). Moreover
2. \( a_k = \frac{f^{(k)}(0)}{k!} \)
3) \( f'(z) = \sum_{k=0}^{\infty} a_k z^{k-1} \). The radius of convergence is also \( R \).

4) \( F(z) = \sum_{k=0}^{\infty} a_k \frac{z^{k+1}}{k+1} \). This series also has radius of convergence \( R \), and \( F'(z) = f(z) \).

5) \( f \) analytic in \( D \), \( z_0 \in D \) Assume \( \{ |z-z_0| < r \} \subset D \) then

\[
f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z-z_0)^k \quad \text{for all } |z-z_0| < r.
\] This is the Taylor series
Examples:
1) \( e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} \) for all \( z \in \mathbb{C} \)
2) \( \frac{1}{1-z} = \sum_{k=0}^{\infty} z^k \) for all \( |z| < 1 \)

Laurent series
\( \{r < |z-z_0| < R\} \subset D \)

If \( f \) is analytic in \( D \), then \( f \) can be expanded as a Laurent series, i.e.
\[
f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k
\]

Example:
1) \( f(z) = \frac{1}{z} \)
2) \( f(z) = \frac{1}{z(1-z)} = \frac{1}{z} (1 + z + z^2 + z^3 + \cdots) \)

\[ = z^{-1} + 1 + z + z^2 + \cdots = \sum_{k=0}^{\infty} z^k \]

\[ k = -1 \]

Fact:

\[ \oint_{\gamma} (z-z_0)^k \, dz = \begin{cases} 0 & k \neq -1 \\ 2\pi i & k = -1 \end{cases} \]

\( \gamma \) is closed, \( z_0 \) inside \( \gamma \), \( \gamma \) does not have self-intersections.

Def: \( f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k \)

\[ a_{-1} = \text{Res} (f, z_0) \text{ is the residue of } f \text{ at } z_0. \]
Example 1) \( f(z) = \frac{1}{(1-z)^2} = 1 + 1 + z + z^2 + \ldots \)

\[ \text{Res} \left( \frac{1}{z(1-z)}, 0 \right) = 1 \]

2) \( f(z) = \frac{1}{(1-z)^2} \quad z_0 = 1 \)

\[ \frac{1}{(1-z)^2} = \frac{(-1)}{(z-1)} \frac{1}{z} = \frac{(-1)}{(z-1)} \left[ \frac{1}{(z-1)+1} \right] \]

\[ \frac{1}{1+(z-1)} = \frac{1}{1-[-1(z+1)]} = 1 + (1)(z-1) + (1)^2(z-1)^2 + (1)^3(z-1)^3 + \ldots \]

\[ \frac{1}{(1-z)^2} = \frac{(-1)}{(z-1)} \sum_{k=0}^{\infty} (-1)^k(z-1)^k \]
\[
\frac{1}{(1-z)^2} = \frac{-1}{(z-1)} + 1 - (z-1) + (z-1)^2 + \ldots
\]

\[
\text{Res} \left( \frac{1}{(1-z)^2}, 1 \right) = -1
\]