Lecture 17

Assume \( f \) is analytic around \( z_0 \) but not defined at \( z_0 \). We say that \( z_0 \) is a singularity. Let

\[
f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \quad \text{Laurent series}
\]

<table>
<thead>
<tr>
<th>Names</th>
<th>Laurent series</th>
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<tbody>
<tr>
<td>Removable singularity</td>
<td>( a_0 + a_1 (z-z_0) + \ldots )</td>
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<tr>
<td>Pole of order ( n )</td>
<td>( a_n (z-z_0)^{-n} + \ldots )</td>
</tr>
<tr>
<td>Simple pole</td>
<td>( a_{-1} (z-z_0)^{-1} + a_0 + \ldots )</td>
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<tr>
<td>Essential singularity</td>
<td>( \ldots + a_{-2} + \frac{a_{-1}}{(z-z_0)^2} + \frac{a_{-2}}{(z-z_0)^3} + \ldots )</td>
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Theorem: If \( f \) and \( g \) are analytic at \( z_0 \), \( f \) has a zero of order \( n \) at \( z_0 \) (\( f(z) = a_n (z-z_0)^n + \ldots, a_n \neq 0 \)), \( g(z_0) \neq 0 \). Then \( F(z) = g(z) / f(z) \) has
a pole of order \( n \) at \( z_0 \)

Example \( F(z) = \frac{2z+5}{(z-1)(z+5)(z-2)^4} \)

1 is a simple pole

-5 is a simple pole

2 is a pole of order 4

Reminder: \( f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \) then

\[ \text{Res} \left( f(z), z_0 \right) = a_{-1} \]

Th: If \( f \) has a simple pole at \( z_0 \)

\[ \text{Res} \left( f(z_0), z_0 \right) = \lim_{z \to z_0} (z-z_0) f(z) \]
Proof: \[ f(z) = \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \ldots. \]

Then \[ (z-z_0)f(z) = a_{-1} + a_0(z-z_0) + a_1(z-z_0)^2 + \ldots. \]

Take \( \lim_{z \to z_0} (z-z_0)f(z) = a_{-1} = \text{Res}(f(z), z_0) \)

Th: If \( f \) has a pole of order \( n \) at \( z_0 \) then

\[ \lim_{z \to z_0} \frac{1}{(z-z_0)^{n-1}} \frac{d^{n-1}}{dz^{n-1}}[(z-z_0)^n f(z)] = \text{Res}(f(z), z_0) \]

Example: \[ f(z) = \frac{1}{(z-1)^2(z-3)} \]

\[ \text{Res}(f(z), 3) = \lim_{z \to 3} (z-3)f(z) = \lim_{z \to 3} \frac{1}{(z-1)^2} = \frac{1}{4} \]
\[
\text{Res} \left( f(z), 1 \right) = \lim_{z \to 1} \frac{1}{\left( z - 1 \right)^2} \frac{d}{dz} f(z) = \lim_{z \to 1} \frac{d}{dz} \frac{1}{z-3} = \\
= -\frac{1}{4} \left| \frac{d}{dz} \right|_{z=1} \]

**Theorem:** If \( D \) is simply connected, \( \gamma \) a simple closed curve in \( D \), \( f \) analytic in \( D \) except at a finite number of singular points \( z_1, \ldots, z_n \) all within \( \gamma \), then

\[
\oint_{\gamma} f(z) \, dz = 2\pi i \sum_{k=1}^{n} \text{Res} (f(z), z_k)
\]
Ex

\[ \oint_C \frac{1}{(z-1)^2(z-3)} \, dz = \]

\[ = 2\pi i \left\{ \text{Res}(f(z), 1) + \text{Res}(f(z), 3) \right\} =
\]

\[ = 2\pi i \left\{ \frac{1}{1!} \frac{d}{dz} \frac{1}{(z-3)} \bigg|_{z=1} + \frac{1}{(z-1)^2} \right\}
\]

\[ = 2\pi i \left\{ \frac{(-1)}{(2-3)^2} \bigg|_{z=1} + \frac{1}{4} \right\} = 0
\]

Integrals of the form \( \int_0^{2\pi} F(\cos \theta, \sin \theta) \, d\theta \)

\[ z = e^{i\theta} \]

\[ \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + z^{-1}}{2} \]

\[ \sin \theta = \frac{z - z^{-1}}{2i} \]
\[ dz = ie^{i\theta} \, d\theta \]

\[ d\theta = \frac{dz}{iz} \]

\[ \int_{0}^{2\pi} F(\cos \theta, \sin \theta) \, d\theta = \oint_{|z| = 1} F \left( \frac{z + z^{-1}}{2} \right) \]

Example: Evaluate \[ \int_{0}^{2\pi} \frac{d\theta}{(2 + \cos \theta)^2} \]

\[ \int_{0}^{2\pi} \frac{d\theta}{(2 + \cos \theta)^2} = \oint_{|z| = 1} \frac{dz}{iz} \left( 2 + \frac{z + z^{-1}}{2} \right)^2 \]

\[ z = e^{i\theta} \]

\[ dz = ie^{i\theta} \, d\theta \]

\[ d\theta = \frac{dz}{iz} \]
\[
\frac{1}{i(2 + \frac{2 + 2^{-1}}{2})^2} = \frac{1}{i(2^{-2})(2 \frac{2}{2} + \frac{2^2 + 1}{2})^2} = \\
\frac{2^2 - 4}{i(2^2 + 42 + 1)^2} = \frac{4}{i(2^2 + 42 + 1)^2} \\
2 = -\frac{4 \pm \sqrt{16 - 4}}{2} = -\frac{4 \pm \sqrt{12}}{2} = -2 \pm \sqrt{3} \\
\]

\[
\text{Res} \left( \frac{4 \frac{2}{2}}{i(2^2 + 42 + 1)^2}, -2 + \sqrt{3} \right) = \\
\]
\begin{align*}
\frac{d}{dz} \frac{4 \ z}{(z^2+4z+3)^2} & \bigg|_{z=-2+\sqrt{3}} = \\
\frac{4}{i} \frac{1}{(z+2+\sqrt{3})^2} - \frac{8}{i} \frac{z}{(z+2+\sqrt{3})^3} & \bigg|_{z=-2+\sqrt{3}} \\
= \frac{4}{i} \frac{1}{12} - \frac{8}{i} \frac{-2+\sqrt{3}}{8(3\sqrt{3})} & = -i \left( \frac{1}{3} - \frac{(-2\sqrt{3} + 3)}{9} \right) = -\frac{i2\sqrt{3}}{9} \\
\int_{0}^{2\pi} \frac{d\theta}{(2+\cos{\theta})^2} & = 2\pi i \left(-\frac{12\sqrt{3}}{9}\right) = \frac{4\sqrt{3}}{9} \pi
\end{align*}
Integrals of the form \( \int_{-\infty}^{\infty} f(x) \, dx = \lim_{R \to \infty} \int_{-R}^{R} f(x) \, dx \)

\( \int_{-R}^{R} f(x) \, dx + \int_{0}^{0} f(x) \, dx \)

**Def:** P.V. \( \int_{-\infty}^{\infty} f(x) \, dx = \lim_{R \to \infty} \int_{-R}^{R} f(x) \, dx \)

**Obs:**

If \( \int_{-\infty}^{\infty} f(x) \, dx \) exists then \( \int_{-\infty}^{\infty} f(x) \, dx = \text{P.V.} \int_{-\infty}^{\infty} f(x) \, dx \)

**Ex:** \( \int_{-\infty}^{\infty} x \, dx = \lim_{R \to \infty} \int_{-R}^{R} x \, dx + \int_{0}^{0} x \, dx \)

\( \int_{-\infty}^{+\infty} x \, dx \) \( \xrightarrow{R \to \infty} -\infty \)

\( \text{P.V.} \int_{-\infty}^{\infty} x \, dx = \lim_{R \to \infty} \int_{-R}^{R} x \, dx = \)

\( \lim_{R \to \infty} 0 = 0 \)

\( \lim_{R \to \infty} \)
\[
\lim_{R \to \infty} \oint_C f(z) \, dz = \lim_{R \to \infty} \int_{-R}^{R} f(x) \, dx \quad \text{if} \quad \lim_{R \to \infty} \oint_{C_2} f(z) \, dz = 0
\]

\[
2\pi i \sum_{k=1}^{n} \text{Res} \left( f(z), z_k \right)
\]

\[z_1, \ldots, z_n\] are the singularities of \( f \) in the upper half of the plane.
\[ \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+9)} = 2\pi i \left\{ \text{Res} \left( \frac{1}{(z^2+1)(z^2+9)}, i \right) + \text{Res} \left( \frac{1}{(z^2+1)(z^2+9)}, 3i \right) \right\} = \]

\[ = 2\pi i \left\{ \frac{1}{2i} \frac{1}{8} + \frac{1}{(-8)} \frac{1}{6i} \right\} = \]

\[ R = \frac{\pi}{4 \left( \frac{1}{2} - \frac{1}{6} \right)} = \frac{\pi}{12} \]

\[ \lim_{R \to \infty} \frac{\pi R}{R^4} = \frac{\pi}{R^3} \to 0 \]
Th: \( f(z) = \frac{P(z)}{Q(z)} \). If degree \( Q \geq \) degree \( P + 2 \)

then

\[ \text{P.V.} \int_{-\infty}^{\infty} f(x) \, dx = \lim_{R \to \infty} \oint_{\mathcal{C}} f(z) \, dz = 2\pi i \sum_{k=1}^{n} \text{Res}(f, z_k) \]

\[ z_1, \ldots, z_n \] singularity of \( f \) in the upper half plane.