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Differential Equations

Def: Unknown is a function. The function and some of its derivatives appear in the equations. If the unknown is a function of one variable, the equation is called an ordinary differential equation (ode)
Example: \( y'' + 3x \cdot y = \cos x \) \hspace{1cm} (1)

Goal: Find \( y = y(x) \) that satisfies

**Def.** The order of an ODE is the order of the highest derivative in the equation.

**Example:** The ODE (1) is of order 2.

**Def.** An ODE is said to be linear if it is of the form

\[
a_n(x) y^{(n)} + a_{n-1}(x) y^{(n-1)} + \cdots + a_1(x) y' + a_0(x) y = g(x)
\]

where \( g, a_0, a_1, \ldots, a_n \) are known functions of \( x \).
Notation: \[ y^{(n)} = \frac{d^n y}{dx^n} \]

Examples:

a) \[ 3xy'' + 6x y = e^x \] is linear
b) \[ (y')^2 + x^2 = 7 \cos y \] is non-linear

Observation: \[ y' = y \quad (2) \]

\[ y = e^x \] and \[ y = 7e^x \] are both solutions of equation (2). Most ODEs have many solutions.

Example:

\[ \begin{cases} x' = 50 \\ x(0) = 5 \end{cases} \]

Then \[ x(t) = 5 + 50t \]. There is only one solution to
(both equations simultaneously).

**IVP, Initial-value problem.**

\[ y^{(n)} = F(x, y, y', \ldots, y^{(n-1)}) \quad \text{ode, } n^{th}\text{ order} \]

\[ y(x_0) = y_0 \]

\[ y'(x_0) = y_1 \]

\[ \vdots \]

\[ y^{(n-1)}(x_0) = y_{n-1} \]

\[ y_0, \ldots, y_{n-1} \text{ are known numbers.} \]

**Fact:** In general, there will be only one solution to these IVPs.
Examples: 1) First order IVP: $\frac{dy}{dx} = f(xy) \quad y(x_0) = y_0$

2) Second order IVP: $y'' = f(x, y, y') \quad y(x_0) = y_0 \quad y'(x_0) = y_1$

Understanding the behavior of the solutions without solving the odes.

Direction fields: This is for first order equations.

Example 1) $y' = \frac{1}{2} xy$.

Draw little segments in the x-y plane with slope $\frac{1}{2} xy$. 
Example: \( y' = \sin y \)
Autonomous ODEs: These are ODEs where the independent variable does not appear explicitly in the ODE.

\[ y'(x) = F(y^{(m-1)}, \ldots, y', y) \]

Example: 1) \( y' = 1 + y^2 \) is 1st order autonomous

2) \( y' = xy \) is not autonomous
Notation: \( \dot{x} = f(x) \), \( \dot{x} = x' = \frac{dx}{dt} \), \( t \) = independent variable

Def: We say \( x_0 \) is a fixed point or a critical point of \( x = f(x) \) if \( f(x_0) = 0 \)

Obs: Let \( x_0 \) be a fixed point of \( x = f(x) \). Then \( x(t) = x_0 \) for all \( t \) is a solution of \( \dot{x} = f(x) \) because \( \dot{x} = 0 \) and

Def: Fixed points are also called equilibrium points.
Stable equilibrium unstable equilibrium

Def. Let $x_0$ be a fixed point of $\dot{x} = f(x)$. We say $x_0$ is stable if $x(t)$ remains close to $x_0$ as long as $x(0)$ was close to $x_0$. Otherwise, we say $x_0$ is unstable.

Solving first order autonomous ODEs

$$\frac{dx}{dt} = f(x) \Rightarrow \frac{dx}{f(x)} = dt \Rightarrow \int \frac{dx}{f(x)} = \int dt$$

Then try to solve for $x$.

Examples: 1) $\dot{x} = kx \quad k$ constant
\[
\frac{dx}{x} = \int k \, dt \quad \Rightarrow \quad \ln |x| = kt + c \quad \Rightarrow \quad |x| = e^{c \ e^{kt}}
\]

\[
A = e^c \quad \text{if } x > 0 \quad x = A \ e^{kt} \quad \text{these are all the solutions}
\]

\[
A = -e^c \quad \text{if } x \leq 0 \quad A \in \mathbb{R}
\]

2) \quad x = x(-x) \quad \Rightarrow \quad \int \frac{dx}{x(-x)} = \int dt

\[
\frac{1}{x(1-x)} = \frac{a}{x} + \frac{b}{(x-1)} = \frac{a(x-1)+bx}{x(x-1)} = \frac{(a+b)x-a}{x(x-1)}
\]

\[
a+b = 0 \quad -a = -1 \quad a = 1 = -1
\]
\[ \int \frac{dx}{x(1-x)} = \int \frac{dx}{x} - \int \frac{dx}{x-1} = \ln |x| - \ln |x-1| = t + C \]

\[ \ln \left| \frac{x}{x-1} \right| = t + C \Rightarrow \left| \frac{x}{x-1} \right| = e^{t+C} = e^t \cdot e^C = A \]

\[ \frac{x}{x-1} = Ae^t, \quad A \in \mathbb{R} \]

\[ x = xAe^t - Ae^t \]

\[ x(1-Ae^t) = -Ae^t \]

\[ x = \frac{-A}{1-Ae^t} \quad \text{or} \quad x = \frac{1}{1-\frac{1}{A}e^{-t}} \]
Phase line: 1st order autonomous ODEs

\[ x' = f(x) \]

Plot \[ y = f(x) \]

- Unstable fixed point
- Stable
- Unstable

The zeros of \( f(x) \) are the fixed points.

Draw arrows in the x-axis, to the right if \( f(x) > 0 \), to the left if \( f(x) < 0 \).

The arrows indicate the direction of motion.
Obs. let \( x_0 \) be a fixed point of \( \dot{x} = f(x) \).

If \( f(x_0) > 0 \) then \( x_0 \) is unstable.

If \( f(x_0) < 0 \) then \( x_0 \) is stable.

If \( f(x_0) = 0 \), we do not know.

Separable equations: \( \frac{dy}{dx} = g(x) \cdot h(y) \)

Separation of variables.
\[ \int \frac{dy}{h(y)} = \int g(x) \, dx \] Then try to solve for \( y \)

**Example:** \( \frac{dy}{dx} = -\frac{x}{y} \) \( y(4) = 3 \)

\[ \int y \, dy = -\int x \, dx \]
\[ \frac{y^2}{2} = -\frac{x^2}{2} + C \]

\[ y = \sqrt{2C - x^2} \] \( y(4) = 3 \implies \)

\[ 3 = \sqrt{2C - 16} \implies y = \sqrt{25 - x^2} \]
Linear first order differential equations

(1) \( y' + P(x) y = f(x) \)

\[ I(x) = e^{\int P(x) \, dx} \]

Then \( I'(x) = P(x) e^{\int P(x) \, dx} \)

\[ I'(x) = P(x) I(x) \]

Multiply eq.(1) by \( I(x) \)

\[ I y' + I P y = I f \]

\[ \frac{d}{dx} (I y) = I f \]

\[ I y = \int I f \]

\[ y = \frac{1}{I(x)} \int I(x) f(x) \, dx \]
Example 1) \( y' - 3y = 6 \) \( \quad y' + Py = f \)

\[ P = -3 \quad I = e^{SP} \]

\[ I = e^{-3x} \]

\[ e^{-3x} y' - 3e^{-3x} y = 6e^{-3x} \]

\[ \frac{d}{dx}(e^{-3x} y) = 6e^{-3x} \]

\[ e^{-3x} y = \int 6e^{-3x} \]

\[ e^{-3x} y = -2e^{-3x} + C \]

\[ y = -2 + Ce^{3x} \]