Trigonometric functions

Obs: \( y \in \mathbb{R} \), then \( e^{iy} = \cos y + i\sin y \)
\( e^{-iy} = \cos y - i\sin y \)

Add & divide by 2 you get \( \cos y = \frac{e^{iy} + e^{-iy}}{2} \)

Subtract & divide by 2i \( \sin y = \frac{e^{iy} - e^{-iy}}{2i} \)

Def. \( z \in \mathbb{C} \) \( \cos z = \frac{e^{iz} + e^{-iz}}{2} \) and \( \sin z = \frac{e^{iz} - e^{-iz}}{2i} \)

Obs: 1) \( \frac{d}{dz} \cos z = \frac{i}{2} e^{iz} - \frac{i}{2} e^{-iz} = (i) \frac{e^{iz} - e^{-iz}}{2i} = -i \sin z \)
2) \( \frac{d}{dz} \sin z = \cos z \)

3) \( \cos^2 z + \sin^2 z = 1 \)

4) \( \cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2 \)

5) \( \sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2 \)

Integrals \( C \) is a curve in the complex plane
\[ \int f = \lim_{n \to \infty} \sum_{k=1}^{n} f(z_k^*) (z_{k+1} - z_k) \]

**Evaluation of integrals**

1) Parameterize \( C \) and find \( z = z(t) \) a function defined on a real interval \( z : [a, b] \to C \)

As \( t \) goes from \( a \) to \( b \) you move in the direction of \( C \) and visit each point exactly once.
\[ \int f = \int_a^b f(z(t)) z'(t) \, dt \]

**Example 1)** \( f \) given by \( x = 3t, \ y = t^2 \) \(-1 \leq t \leq 4\)

\[ z(t) = 3t + it^2 \quad -1 \leq t \leq 4 \]

Compute \( \int \overline{z} \, z' \, dt \)

\[ \int_{-1}^{4} (3t - it^2)(3 + 2it) \, dt = \]

\[ = \int_{-1}^{4} (9t + 2t^3) + i(6t^2 - 3t^2) \, dt = \left( \frac{9t^2 + t^4}{2} \right)_{-1}^{4} + i \left( \frac{t^3}{3} \right)_{-1}^{4} = \]
\[
\left( \frac{9}{2} \cdot 4^2 + \frac{4^4}{2} \right) - \left( \frac{9}{2} + \frac{1}{2} \right) + i \left( 4^3 - (-1)^3 \right)
\]

Ex: \( C \): circle centered at 0 of radius 1 in the counter-clockwise direction. Compute \( \int_{C} \frac{1}{z} \, dz = \int_{0}^{2\pi} \frac{-\text{e}^{it} \, dt}{z} \) \( \text{e}^{it} \)

\[
\int_{0}^{2\pi} \frac{-\text{e}^{it} \, dt}{\text{e}^{it}} = 2\pi i
\]

Obs: The integral does not depend on the parametrization.

Properties: 1) \( \oint_{C} c f(z) \, dz = c \oint_{C} f(z) \, dz \) \( c \in \mathbb{C} \)
2) \[ \int_{\gamma} (f + g)(z) \, dz = \int_{\gamma_1} f(z) \, dz + \int_{\gamma_2} g(z) \, dz \]

3) \[ \int_{\gamma} f(z) \, dz = \int_{\gamma_1} f(z) \, dz + \int_{\gamma_2} f(z) \, dz \]
\[ \gamma = \gamma_1 \cup \gamma_2 \]

Notation \( \gamma = \gamma_1 + \gamma_2 \)

4) \[ -\gamma \]
\[ -\gamma \text{ is } \gamma \text{ but traveled in the opposite direction} \]
\[ \int_{-\gamma} f(z) \, dz = -\int_{\gamma} f(z) \, dz \]
Def. Connected sets

Connected

Not connected

Def. A set $D$ is simply connected if it is connected and does not have holes.

Ex. a) Simply connected

b) Not simply connected
c) not simply connected (a point is missing) inside

Theorem: \( D \) simply connected. \( \gamma \) closed curve, \( \gamma \subset D \)

\[ \text{D open, } f \text{ analytic in } D. \]

Then:

Closed curves

Not closed

\[ \int_{C} f(z) \, dz = 0 \]
Examples: 1) $f(z) = z$
$$\int_{0}^{2\pi} e^{it} \, dt = i \left. \frac{1}{2i} e^{2it} \right|_{0}^{2\pi} = 0$$

2) Same curve, but $f(z) = \frac{1}{z}$
$$\int_{0}^{2\pi} \frac{1}{z} \, dz = 2\pi i \quad \text{OK, because} \quad \frac{1}{z} \quad \text{is not differentiable at} \quad z = 0.$$
Obs

\[ \mathcal{L}_1 \text{ & } \mathcal{L}_2 \text{ closed curves} \]

\[ \mathcal{L}_1 \text{ without self intersections.} \]

\[ \mathcal{L}_2 \text{ inside } \mathcal{L}_1 \text{ (they do not intersect)} \]

Yellow region = outside \( \mathcal{L}_2 \) & inside \( \mathcal{L}_1 \)

+ analytic on \( \mathcal{L}_1 \), on \( \mathcal{L}_2 \), and between the curves (in the yellow region).

\[ \mathcal{L}_4 = - \mathcal{L}_3 \]

\[ 0 = \int_{\mathcal{L}_1} f = \int_{\mathcal{L}_2} f + \int_{\mathcal{L}_3} f + \int_{\mathcal{L}_4} f = \int_{\mathcal{L}_2} f - \int_{\mathcal{L}_3} f - \int_{\mathcal{L}_4} f \]
\[ f = f_2 + f_3 - f_1 - f_3 = \int_{f_2} f + \int_{f_3} f - \int_{f_1} f - \int_{f_3} f \]

\[ 0 = \int_{f_2} f + \int_{f_1} f \]

\[ \oint_{f_1} f = \oint_{f_2} f \]

**Example**

\[ \int_{\frac{1}{z}} \frac{1}{z} \, dz = \int_{\frac{1}{z}} \frac{1}{z} \, dz = 2\pi i \]

\[ f : |z| = r \]
The only point $z^* = \chi$ where $f$ is not analytic

\[
\int_{\gamma} f = \lim_{\varepsilon \to 0} \int_{|z-z^*| = \varepsilon} f
\]

Observations:

$\gamma$ analytic inside $\gamma$ and $\gamma \setminus \gamma_1 \cup \gamma_2$ outside $\gamma_1$ and $\gamma_2$.

Then

\[
\int_{\gamma} f = \int_{\gamma_1} f + \int_{\gamma_2} f + \int_{\gamma_1} f
\]
Example

\[ \oint \frac{dz}{z^2 + 1} \]

\[ |z| = 3 \]

\[ z^2 + 1 = 0 \quad \text{then} \quad z = \pm i \]

\[ \oint_{|z| = 3} \frac{dz}{z^2 + 1} = \oint_{|z-1| = 3} \frac{dz}{z^2 + 1} + \oint_{|z+1| = 3} \frac{dz}{z^2 + 1} \]

\[
\int_{|z-i|=3} \frac{dz}{z^2 + 1} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( \frac{1}{i} e^{i\theta} + \frac{1}{-i} e^{-i\theta} \right) d\theta
\]

\[ z = i + e^{i\theta} \]
\[
\int_0^{2\pi} \frac{e^{it} e^{it}}{2(e^{it} + g e^{2it})} \, dt = \int_0^{2\pi} \frac{i}{2i + \epsilon e^{it}} \, dt =
\]

\[
= \int_0^{2\pi} \frac{i}{2i} \, dt = \pi
\]

Verify
\[
\oint_{|z| = 3} \frac{dz}{z^2 + 1} = -\pi
\]

\[
\oint_{|z| = 3} \frac{dz}{z^2 + 1} = 0
\]
Obs: \( f \) analytic in \( D \), \( D \) open. 
\( D \) simply connected.

\( \gamma_1 \) and \( \gamma_2 \) curves in \( D \)

\[ 0 = \int_{\gamma_1 - \gamma_2} f = \int_{\gamma_1} f - \int_{\gamma_2} f \]

\[ \int_{\gamma_1} f = \int_{\gamma_2} f \]
Obs: If open simply connected, $f$ analytic in $D$. Then $\int_{C} f$ depends only on the end points of $C$.

Ex:

\[
\int_{\ell_1} \frac{1}{z} \, dz = \pi i \\
\int_{\ell_2} \frac{1}{z} \, dz = -\pi i
\]

not equal

not contradiction of theorem because $\frac{1}{z}$ is not
analytic at 0 and 0 is inside the region determined
by the two curves.

**Theorem.** D open simply connected. f analytic in D. then
there exists F in D such that \( F(z) = f(z) \) for all \( z \in D \). If \( z_1 \) a curve in D, \( z_i \) its initial and \( z_f \) its
final points. Then

\[
\int_{z_1}^{z_f} f = F(z_f) - F(z_i)
\]
Cauchy's integral formula

If \( D \) is simply connected open, \( f \) analytic in \( D \), \( \gamma \subset D \)
\( \gamma \) closed curve without self intersections, \( z_0 \) inside \( \gamma \).

Then \( f(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-z_0} \, dz \).

Proof: \( g(z) = \frac{f(z)}{z-z_0} \)
\( g(z_0) = 0 \) = \( 0 \) to \( z \to 0 \) to get the result.