Integration on Manifolds, Volume, and Partitions of Unity

Suppose that we have an orientable Riemannian manifold \((M,g)\) and a function \(f: M \to \mathbb{R}\). How can we define the integral of \(f\) on \(M\)? First we answer this question locally, i.e., if \((U,\phi)\) is a chart of \(M\) (which preserves the orientation of \(M\)), we define

\[
\int_U f \, dv_g := \int_{\phi(U)} f(\phi^{-1}(x)) \sqrt{\det(g^\phi_{ij}(\phi^{-1}(x)))} \, dx,
\]

where \(g_{ij}\) are the coefficients of the metric \(g\) in local coordinates \((U,\phi)\). Recall that \(g^\phi_{ij}(p) := g(E^\phi_i(p), E^\phi_j(p))\), where \(E^\phi_i(p) := d\phi^{-1}_p(e_i)\).

Now note that if \((V,\psi)\) is any other (orientation preserving) local chart of \(M\), and \(W := U \cap V\), then there are two ways to compute \(\int_W f \, dv_g\), and for these to yield the same answer we need to have

\[
\int_{\phi(W)} f(\phi^{-1}(x)) \sqrt{\det(g^\phi_{ij}(\phi^{-1}(x)))} dx = \int_{\psi(W)} f(\psi^{-1}(x)) \sqrt{\det(g^\psi_{ij}(\psi^{-1}(x)))} dx.
\]

To check whether the above expression is valid recall that the change variables formula tells that if \(D \subset \mathbb{R}^n\) is an open subset, \(f: D \to \mathbb{R}\) is some function, and \(u: \tilde{D} \to D\) is a diffeomorphism, then

\[
\int_D f(x) \, dx = \int_{\tilde{D}} f(u(x)) \det(du_x) \, dx.
\]

Now recall that, by the definition of manifolds, \(\phi \circ \psi^{-1}: \psi(W) \to \phi(W)\) is a diffeomorphism. So, by the change of variables formula, the integral on the left hand side of (1) may be rewritten as

\[
\int_{\psi(W)} f(\psi^{-1}(x)) \sqrt{\det(g^\psi_{ij}(\psi^{-1}(x)))} \det(d(\phi \circ \psi)^{-1}_x) dx.
\]
So for equality in (1) to hold we just need to check that
\[ \sqrt{\det(g_{ij}^\psi(\psi^{-1}(x)))} = \sqrt{\det(g_{ij}^\phi(\psi^{-1}(x))) \det(d(\phi \circ \psi^{-1})_x)}, \]
for all \( x \in \psi(W) \) or, equivalently,
\[ \sqrt{\det(g_{ij}^\psi(p))} = \sqrt{\det(g_{ij}^\phi(p)) \det(d(\phi \circ \psi^{-1})_{\psi(p)})}, \tag{2} \]
for all \( p \in W \). To see that the above equality holds, let \( (a_{ij}) \) be the matrix of the linear transformation \( d(\phi \circ \psi^{-1}) \) and note that
\[
\begin{align*}
g_{ij}^\psi &= g(d\psi^{-1}(e_i), d\psi^{-1}(e_j)) \\
&= g(d\phi^{-1} \circ d(\phi \circ \psi^{-1})(e_i), d\phi^{-1} \circ d(\phi \circ \psi^{-1})(e_j)) \\
&= g \left( d\phi^{-1} \left( \sum_{\ell} a_{i\ell} e_\ell \right), d\phi^{-1} \left( \sum_k a_{jk} e_k \right) \right) \\
&= \sum_{\ell k} a_{i\ell} a_{jk} g_{\ell k}^\phi.
\end{align*}
\]
So if \( (g_{ij}^\psi) \) and \( (g_{ij}^\phi) \) denote the matrices with the coefficients \( g_{ij}^\psi \) and \( g_{ij}^\phi \), then we have
\[
(g_{ij}^\psi) = (a_{ij})(a_{ij})(g_{ij}^\phi).
\]
Taking the determinant of both sides of the above equality yields (2). In particular note that \( \sqrt{\det(a_{ij})^2} = |\det(a_{ij})| = \det(a_{ij}) \), because, since \( M \) is orientable and \( \phi \) and \( \psi \) are by assumption orientation preserving charts, \( \det(a_{ij}) > 0 \).

Next we discuss, how to integrate a function on all of \( M \). To see this we need the notion of *partition of unity* which may be defined as follows: Let \( U_i, i \in I, \) be an open cover of \( M \), then by a (smooth) partition of unity subordinate to \( U_i \) we mean a collection of smooth functions \( \theta_i: M \to \mathbb{R} \) with the following properties:

1. \( \text{supp} \theta_i \subset A_i. \)
2. for any \( p \in M \) there exists only finitely many \( i \in I \) such that \( \theta_i(p) \neq 0. \)
3. \( \sum_{i \in I} \theta_i(p) = 1, \) for all \( p \in M. \)

Here \( \text{supp} \) denotes *support*, i.e., the closure of the set of points where a given function is nonzero. Further note that by property 2 above, the sum in item 3 is well-defined.

**Theorem 0.1.** If \( M \) is any smooth manifold, then any open covering of \( M \) admits a subordinate smooth partition of unity.
Using the above theorem, whose proof we postpone for the time being, we may define \( \int_M f dv_g \), for any function \( f : M \to \mathbb{R} \) as follows. Cover \( M \) by a family of local charts \( (U_i, \phi_i) \), and let \( \theta_i \) be a subordinate partition of unity. Then we set
\[
\int_M f dv_g := \sum_{i \in I} \int_{U_i} \theta_i f dv_g.
\]
Note that this definition does not depend on the choice of local charts or the corresponding partitions of unity. The *volume* of any orientable Riemannian manifold may now be defined as the integral of the constant function one:
\[
\text{vol}(M) := \int_M dv_g.
\]
Now we proceed towards proving Theorem 0.1.

**Exercise 0.2.** Compute the area of a torus of revolution in \( \mathbb{R}^3 \).

**Lemma 0.3.** Any open cover of a manifold has a countable subcover.

*Proof.* Suppose that \( U_i, i \in I \), is an open covering of a manifold \( M \) (where \( I \) is an arbitrary set). By definition, \( M \) has a countable basis \( B = \{B_j\}_{j \in J} \). For every \( i \in I \), let \( A_i := \{B_j \mid B_j \subset U_i\} \). Then \( A_i \) is an open covering for \( M \). Next, let \( A := \bigcup_{i \in I} A_i \). Since \( A \subset B \), \( A \) is countable, so we may denote the elements of \( A \) by \( A_k \), where \( k = 1, 2, \ldots \). Note that \( A_k \) is still an open covering for \( M \). Further, for each \( k \) there exists an \( i \in I \) such that \( A_k \subset U_i \). We may collect all such \( U_i \) and reindex them by \( k \), which gives the desired countable subcover. \( \square \)

**Lemma 0.4.** Any manifold has a countable basis such that each basis element has compact closure.

*Proof.* By the previous lemma we may cover any manifold \( M \) by a countable collection of charts \( (U_i, \phi_i) \). Let \( V_j \) be a countable basis of \( \mathbb{R}^n \) such that each \( V_j \) has compact closure \( \overline{V_j} \), e.g., let \( V_j \) be the set of balls in \( \mathbb{R}^n \) centered at rational points and with rational radii less than 1. Then \( B_{ij} := \phi_i^{-1}(V_j) \) gives a countable basis for \( U_i \) such that each basis element has compact closure, since \( \overline{B_{ij}} = \phi_i^{-1}(\overline{V_j}) \). So \( \bigcup_{ij} B_{ij} \) gives the desired collection, since a countable collection of countable sets is countable. \( \square \)

**Lemma 0.5.** Any manifold \( M \) is countable at infinity, i.e., there exists a countable collection of compact subsets \( K_i \) of \( M \) such that \( M \subset \bigcup_i K_i \) and \( K_i \subset \text{int} K_{i+1} \).

*Proof.* Let \( B_i \) be the countable basis of \( M \) given by the previous lemma, i.e., with each \( \overline{B_i} \) compact. Set \( K_1 := \overline{B_1} \) and let \( K_{i+1} := \bigcup_{j=1}^r B_j \), where \( r \) is the smallest integer such that \( K_i \subset \bigcup_{j=1}^r B_j \). \( \square \)
By a refinement of an open cover \( U_i \) of \( M \) we mean an open cover \( V_j \) such that for each \( j \in J \) there exists \( i \in I \) with \( V_j \subset U_i \). We say that an open covering is locally finite, if for every \( p \in M \) there exists finitely many elements of that covering which contain \( p \).

**Lemma 0.6.** Any open covering of a manifold \( M \) has a countable locally finite refinement by charts \((U_i, \phi_i)\) such that \( \phi_i(U_i) = B^n(o) \) and \( V_i := \phi^{-1}(B^n(o)) \) also cover \( M \).

**Proof.** First note that for every point \( p \in M \), we may find a local chart \((U_p, \phi_p)\) such that \( \phi_p(U_p) = B^n(o) \), and set \( V_p := \phi^{-1}(B^n(o)) \). Further, we may require that \( U_p \) lies inside any given open set which contains \( p \). Let \( A_\alpha \) be an open covering for \( M \). By a previous lemma, after replacing \( A_\alpha \) by a subcover, we may assume that \( A_\alpha \) is countable. Now consider the sets \( A_\alpha \cap (\text{int} \ K_{i+2} - K_{i-1}) \). Since \( K_{i+1} - \text{int} \ K_i \) is compact, there exists a finite number of open sets \( U_{p_j}^{\alpha,i} \subset A_\alpha \cap (\text{int} \ K_{i+2} - K_{i-1}) \) such that \( V_{p_j}^{\alpha,i} \) covers \( A_\alpha \cap (K_{i+1} - \text{int} \ K_i) \). Since \( K_i \) and \( A_\alpha \) are countable, the collection \( U_{p_j}^{\alpha,i} \) is a countable. Further, by construction \( U_{p_j}^{\alpha,i} \) is locally finite, so it is the desired refinement. \( \square \)

**Note 0.7.** The last result shows in particular that every manifold is paracompact, i.e., every open cover of \( M \) has a locally finite refinement.

**Proof of Theorem 0.1.** Let \( A_\alpha \) be an open cover of \( M \). Note that if \( U_i \) is any refinement of \( A_\alpha \) and \( \theta_i \) is a partition of unity subordinate to \( U_i \) then, \( \theta_i \) is subordinate to \( A_\alpha \). In particular, it is enough to show that the refinement \( U_i \) given by the previous lemma has a subordinate partition of unity. To this end note that there exists a smooth nonnegative function \( f: \mathbb{R} \to \mathbb{R} \) such that \( f(x) = 0 \) for \( x \geq 2 \), and \( f(x) = 1 \) for \( x \leq 1 \). Define \( \overline{\theta}_i : M \to \mathbb{R} \) by \( \overline{\theta}_i(p) := f(||\phi_i(p)||) \) if \( p \in U_i \) and \( \overline{\theta}_i(p) := 0 \) otherwise. Then \( \overline{\theta}_i \) are smooth. Finally, \( \theta_i(p) := \overline{\theta}_i(p)/\sum_j \overline{\theta}_j(p) \), is the desired partition of unity. \( \square \)

Recall that earlier we showed that any compact manifold admits a Riemannian metric, since it can be isometrically embedded in some Euclidean space. As an application of the previous result we now can show:

**Corollary 0.8.** Any manifold admits a Riemannian metric

**Proof.** Let \((U_i, \phi_i)\) be an atlas of \( M \), and let \( \theta_i \) be a subordinate partition of unity. Now for \( p \in U_i \) define \( g_p(X, Y) := \langle d\phi_i(X), d\phi_i(Y) \rangle \). Then we define a Riemannian metric \( g \) on \( M \) by setting \( g_p(X, Y) := \sum_i \theta_i(p) g_p(X, Y) \). \( \square \)

**Exercise 0.9.** Show that every manifold is normal, i.e., for every disjoint closed sets \( A_1, A_2 \) in \( M \) there exists a pair of disjoint open subsets \( U_1, U_2 \) of \( M \) such that \( X_1 \subset U_1 \) and \( X_2 \subset U_2 \). [Hint: Use the fact that every manifold admits a metric]
Exercise 0.10. Show that if $U$ is any open subset of a manifold $M$ and $A \subset U$ is a closed subset, then there exists smooth function $f: M \to \mathbb{R}$ such that $f = 1$ on $A$ and $f = 0$ on $M - U$.

Exercise 0.11. Compute the volume (area) of a torus of revolution in $\mathbb{R}^3$.

Exercise 0.12. Let $M \subset \mathbb{R}^n$ be an embedded submanifold which may be parameterized by $f: U \to \mathbb{R}^n$, for some open set $U \subset \mathbb{R}^m$, i.e., $f$ is a one-to-one smooth immersion and $f(U) = M$. Show that then $\text{vol}(M) = \int_U \sqrt{\det(J_x(f) \cdot J_x(f)^T)} \, dx$, where $J_x(f)$ is the jacobian matrix of $f$ at $x$. 