

2.5 The inverse function theorem

Recall that if \( f : M \to N \) is a diffeomorphism, then \( df_p \) is nonsingular at all \( p \in M \) (by the chain rule and the observation that \( f \circ f^{-1} \) is the identity function on \( M \)). The main aim of this section is to prove a converse of this phenomenon:

**Theorem 1** (The Inverse Function Theorem). Let \( f : M \to N \) be a smooth map, and \( \dim(M) = \dim(N) \). Suppose that \( df_p \) is nonsingular at some \( p \in M \). Then \( f \) is a local diffeomorphism at \( p \), i.e., there exists an open neighborhood \( U \) of \( p \) such that

1. \( f \) is one-to-one on \( U \).
2. \( f(U) \) is open in \( N \).
3. \( f^{-1} : f(U) \to U \) is smooth.

In particular, \( d(f^{-1})_{f(p)} = (df_p)^{-1} \).

A simple fact which is applied a number of times in the proof of the above theorem is

**Lemma 2.** Let \( f : M \to N \), and \( g : N \to L \) be diffeomorphisms, and set \( h := g \circ f \). If any two of the mappings \( f, g, h \) are diffeomorphisms, then so is the third. \( \square \)

In particular, the above lemma implies

**Proposition 3.** If Theorem 1 is true in the case of \( M = \mathbb{R}^n = N \), then, it is true in general.
Proof. Suppose that Theorem 1 is true in the case that \( M = \mathbb{R}^n = N \), and let \( f: M \to N \) be a smooth map with \( df_p \) nonsingular at some \( p \in M \). By definition, there exist local charts \((U, \phi)\) of \( M \) and \((V, \psi)\) of \( N \), centered at \( p \) and \( f(p) \) respectively, such that \( \tilde{f} := \phi^{-1} \circ f \circ \psi \) is smooth. Since \( \phi \) and \( \psi \) are diffeomorphisms, \( d\phi_p \) and \( d\psi_{f(p)} \) are nonsingular. Consequently, by the chain rule, \( d\tilde{f}_o \) is nonsingular, and is thus a local diffeomorphism. More explicitly, there exists open neighborhoods \( A \) and \( B \) of the origin \( o \) of \( \mathbb{R}^n \) such that \( \tilde{f}: A \to B \) is a diffeomorphism. Since \( \phi: \phi^{-1}(A) \to A \) is also a diffeomorphism, it follows that \( \phi \circ \tilde{f} : \phi^{-1}(A) \to B \) is a diffeomorphism. But \( \phi \circ \tilde{f} = f \circ \psi \). So \( f \circ \psi : \phi^{-1}(A) \to B \) is a diffeomorphism. Finally, since \( \psi: \psi^{-1}(B) \to B \) is a diffeomorphism, it follows, by the above lemma, that \( f: \phi^{-1}(A) \to \psi^{-1}(B) \) is a diffeomorphism. \( \square \)

So it remains to prove Theorem 1 in the case that \( M = \mathbb{R}^n = N \). To this end we need the following fact. Recall that a metric space is said to be complete provided that every Cauchy sequence of that space converges.

**Lemma 4** (The contraction Lemma). Let \( (X,d) \) be a complete metric space, and \( 0 \leq \lambda < 1 \). Suppose that there exists mapping \( f: X \to X \) such that \( d(f(x_1), (x_2)) \leq \lambda d(x_1, x_2) \), for all \( x_1, x_2 \in X \). Then there exists a unique point \( x \in X \) such that \( f(x) = x \).

**Proof.** Pick a point \( x_0 \in X \) and set \( x_n := f^n(x) \), for \( n \geq 1 \). We claim that \( \{x_n\} \) is a Cauchy sequence. To this end note that

\[
d(x_n, x_{n+m}) = d(f^n(x_0), f^n(x_m)) \leq \lambda^n d(x_0, x_m).
\]

Further, by the triangle inequality

\[
d(x_0, x_m) \leq d(x_0, x_1) + d(x_1, x_2) + \cdots + d(x_{m-1}, x_m)
\leq (1 + \lambda + \lambda^2 + \cdots + \lambda^m)d(x_0, x_1)
\leq \frac{1}{1-\lambda}d(x_0, x_1).
\]

So, setting \( K := d(x_0, x_1)/(1-\lambda) \), we have

\[
d(x_n, x_{n+m}) \leq \lambda^n K.
\]

Since \( K \) does not depend on \( m \) or \( n \), the last inequality shows that \( \{x_n\} \) is a Cauchy sequence, and therefore, since \( X \) is complete, it has a limit point, say \( x_\infty \). Now note that, since \( d: X \times X \to \mathbb{R} \) is continuous (why?),

\[
d(x_\infty, f(x_\infty)) = \lim_{n \to \infty} d(x_n, f(x_n)) = 0.
\]
Thus $X_\infty$ is a fixed point of $f$. Finally, note that if $a$ and $b$ are fixed points of $f$, then

$$d(a, b) = d(f(a), f(b)) \leq \lambda d(a, b),$$

which, since $\lambda < 1$, implies that $d(a, b) = 0$. So $f$ has a unique fixed point.

**Exercise 5.** Does the previous lemma remain valid if the condition that $d(f(x_1), (x_2)) \leq \lambda d(x_1, x_2)$ is weakened to $d(f(x_1), (x_2)) < d(x_1, x_2)$?

Next we recall

**Lemma 6** (The mean value theorem). Let $f : \mathbb{R}^n \to \mathbb{R}$ be a $C^1$ functions. Then for every $p, q \in \mathbb{R}^n$ there exists a point $s$ on the line segment connecting $p$ and $q$ such that

$$f(p) - f(q) = Df(s)(p - q) = \sum_{i=1}^{n} D_{ij}f(s_i)(p^i - q^i).$$

**Exercise 7.** Prove the last lemma by using the mean value theorem for functions of one variable an the chain rule. *(Hint: Parametrize the segment joining $p$ and $q$ by $tq + (1 - t)p$, $0 \leq t \leq 1$).*

The above lemma implies:

**Proposition 8.** Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a $C^1$ function, $U$ be a convex open neighborhood of $0$ in $\mathbb{R}^n$, and set

$$K := \sup \left\{ \|D_{ij}f^i(p)\| \mid 1 \leq i \leq m, 1 \leq j \leq n, p \in U \right\}$$

Then, for every $p, q \in U$,

$$\|f(p) - f(q)\| \leq \sqrt{mn}K\|p - q\|$$

**Proof.** First note that

$$\|f(p) - f(q)\|^2 = \sum_{i=1}^{m} (f^i(p) - f^i(q))^2.$$
Secondly, by the mean value theorem (Lemma 6), there exists, for every \( i \) a point \( s_i \) on the line segment connecting \( p \) and \( q \) such that

\[
f^i(p) - f^i(q) = Df^i(s_i)(p - q) = \sum_{j=1}^{n} D_j f^i(s_j)(p - q).
\]

Since \( U \) is convex, \( s_i \in U \), and, therefore, by the Cauchy-Schwartz inequality

\[
|f^i(p) - f^i(q)| \leq \sqrt{\sum_{j=1}^{n} D_j f^i(s_j)^2} \cdot \sqrt{\sum_{j=1}^{n} (p^j - q^j)^2} \leq \sqrt{n}K\|p - q\|.
\]

So we conclude that

\[
\|f(p) - f(q)\|^2 \leq mnK^2\|p - q\|^2.
\]

Finally, we recall the following basic fact

**Lemma 9.** Let \( f : \mathbb{R}^n \rightarrow \mathbb{R}^m \), and \( p \in \mathbb{R}^n \). Suppose there exists a linear transformation \( A : \mathbb{R}^n \rightarrow \mathbb{R}^m \) such that

\[
f(x) - f(p) = A(p - x) + r(x, p)
\]

where \( r : \mathbb{R}^2 \rightarrow \mathbb{R} \) is a function satisfying

\[
\lim_{x \to p} \frac{r(x, p)}{\|x - p\|} = 0.
\]

Then all the partial derivatives of \( f \) exist at \( p \), and \( A \) is given by the jacobian matrix \( D f(p) := (D_1 f(p), \ldots, D_n f(p)) \) whose columns are the partial derivatives of \( f \). In particular, \( A \) is unique. Conversely, if all the partial derivative \( D_i f(p) \) exist, then \( A := D f(p) \) satisfies the above equation.

**Proof.** Let \( e_1, \ldots, e_n \) be the standard basis for \( \mathbb{R}^n \). Then

\[
D_i f(p) = \lim_{t \to 0} \frac{f(p + te_i) - f(p)}{t} = \lim_{t \to 0} \frac{A(te_i) + r(p + te_i, p)}{t} = A(e_i).
\]

Thus all the partial derivatives of \( f \) exist at \( p \), and \( D_i f(p) \) coincides with the \( i^{th} \) column of (the matrix representation) of \( A \). In particular, \( A = D f(p) \) and therefore \( A \) is unique.
Conversely, suppose that all the partial derivatives $D_i f(p)$ exist and set
\[ r(x, p) := f(x) - f(p) - Df(p)(p - x). \]
By the mean value theorem,
\[ r(x, p) = (Df(s) - Df(p))(p - x) \]
for some $s$ on the line segment joining $p$ and $s$. Thus it follows that
\[ \lim_{x \to p} \frac{r(x, p)}{\|x - p\|} = \lim_{x \to p} (Df(s) - Df(p)) \left( \frac{p - x}{\|p - x\|} \right) = 0, \]
as desired. \hfill \Box

Now we are finally ready to prove the main result of this section.

Proof of Theorem 1. By 3 we may assume that $M = \mathbb{R}^n = N$. Further, after replacing $f(x)$ with $(Df(p))^{-1}f(x - p) - f(p)$ we may assume, via Lemma 2, that
\[ p = o, \quad f(o) = o, \quad \text{and} \quad Df(o) = I, \]
where $I$ denotes the identity matrix. Now define $g: \mathbb{R}^n \to \mathbb{R}^n$ by
\[ g(x) = x - f(x). \]
Then $g(o) = o$, and $Dg(o) = 0$. Thus, by Proposition 8, there exists $r > 0$ such that for all $x_1, x_2 \in B_r(o)$, the closed ball of radius $r$ centered at $o$,
\[ \|g(x_1) - g(x_2)\| \leq \frac{1}{2} \|x_1 - x_2\|. \]
In particular, $\|g(x)\| = \|g(x) - g(o)\| \leq \|x\|/2$. So $g(B_r(o)) \subset B_{r/2}(o)$. Now, for every $y \in B_{r/2}(o)$ and $x \in B_r(o)$ define
\[ T_y(x) := y + g(x) = y + x - f(x). \]
Then, by the triangle inequality, $\|T_y(x)\| \leq r$. Thus $T_y: B_r(o) \to B_r(o)$. Further note that
\[ T_y(x) = x \iff y = f(x). \]
in particular, \( T_y \) has a unique fixed point on \( B_r(o) \) if and only if \( f \) is one-to-one on \( B_r(o) \). But 

\[
\|T_y(x_1) - T_y(x_2)\| = \|g(x_1) - g(x_2)\| \leq \frac{1}{2} \|x_1 - x_2\|.
\]

Thus by Lemma 4, \( T_y \) does indeed have a unique fixed point, and we conclude that \( f \) is one-to-one on \( B_r(o) \). In particular, we let \( U \) be the interior of \( B_r(o) \).

Next we show that \( f(U) \) is open. To this end it suffices to prove that \( f^{-1}: f(B_r(o)) \to B_r(o) \) is continuous. To see this note that, by the definition of \( g \) and the triangle inequality,

\[
\|g(x_1) - g(x_2)\| = \|(x_1 - x_2) - (f(x_1) - f(x_2))\| \geq \|x_1 - x_2\| - \|f(x_1) - f(x_2)\|.
\]

Thus,

\[
\|f(x_1) - f(x_2)\| \geq \|x_1 - x_2\| - \|g(x_1) - g(x_2)\| = \frac{1}{2} \|x_1 - x_2\|,
\]

which in turn implies

\[
\|y_1 - y_2\| \geq \frac{1}{2} \|f^{-1}(y_1) - f^{-1}(y_2)\|.
\]

So \( f^{-1} \) is continuous.

It remains to show that \( f^{-1} \) is smooth on \( f(U) \). To this end, note that by Lemma 9, for every \( p \in U \),

\[
f(x) - f(p) = Df(p)(x - p) + r(x, p).
\]

Now multiply both sides of the above equality by \( A := (Df(p))^{-1} \), and set \( y := f(x), q := f(p) \). Then

\[
A(y - q) = f^{-1}(y) - f^{-1}(q) + Ar(f^{-1}(y), f^{-1}(q)),
\]

which we may rewrite as

\[
f^{-1}(y) - f^{-1}(q) = A(y - q) + \overline{r}(y, q),
\]

where

\[
\overline{r}(y, q) := Ar(f^{-1}(y), f^{-1}(q)).
\]
Finally note that
\[
\lim_{y \to q} \frac{r(y, q)}{\|y - q\|} = A \lim_{y \to q} \frac{r(f^{-1}(y), f^{-1}(q))}{\|y - q\|} \leq 2A \lim_{y \to q} \frac{r(f^{-1}(y), f^{-1}(q))}{\|f^{-1}(y) - f^{-1}(q)\|} = 0.
\]
Thus, again by Lemma 9, \( f^{-1} \) is differentiable at all \( p \in U \) and
\[
D(f^{-1})(p) = \left( Df(f^{-1}(p)) \right)^{-1}.
\]
Since the right hand side of the above equation is a continuous function of \( p \) (because \( f \) is \( C^1 \) and \( f^{-1} \) is continuous), it follows that \( f^{-1} \) is \( C^1 \). But if \( f \) is \( C^r \), then the right hand side of the above equation is \( C^r \) (since \( Df \) is \( C^\infty \) everywhere), which in turn yields that \( f^{-1} \) is \( C^{r+1} \). So, by induction, \( f^{-1} \) is \( C^\infty \).

**Exercise 10.** Give a simpler proof of the inverse function theorem for the special case of mappings \( f: \mathbb{R} \to \mathbb{R} \).

### 2.6 The rank theorem

The inverse function theorem we proved in the last section yields the following more general result:

**Theorem 11** (The rank theorem). Let \( f: M \to N \) be a smooth map, and suppose that \( \text{rank}(df_p) = k \) for all \( p \in M \), then, for each \( p \in M \), there exists local charts \((U, \phi)\) and \((V, \psi)\) of \( M \) and \( N \) centered at \( p \) and \( f(p) \) respectively such that
\[
\psi \circ f \circ \phi^{-1}(x_1, \ldots, x_n) = (x_1, \ldots, x_k, 0, \ldots, 0).
\]

**Exercise 12.** Show that to prove the above theorem it suffices to consider the case \( M = \mathbb{R}^n \) and \( N = \mathbb{R}^m \). Furthermore, show that we may assume that \( p = o, f(o) = o \), and the \( k \times k \) matrix in the upper left corner of the jacobian matrix \( Df(o) \) is nonsingular.

**Proof.** Suppose that the conditions of the previous exercise hold. Define \( \phi: \mathbb{R}^n \to \mathbb{R}^n \) by
\[
\phi(x) := (f^1(x), \ldots, f^k(x), x^{k+1}, \ldots, x^n).
\]
Then
\[
D\phi(o) = \begin{pmatrix}
\frac{\partial(f^1, \ldots, f^k)}{(x^1, \ldots, x^k)}(o) & * \\
0 & I_{n-k}
\end{pmatrix}.
\]
Thus $D\phi(o)$ is nonsingular. So, by the inverse function theorem, $\phi$ is a local diffeomorphism at $o$. In particular $\phi^{-1}$ is well defined on some open neighborhood $U$ of $o$. Let $\pi_i: \mathbb{R}^k \to \mathbb{R}$ be the projection onto the $i^{th}$ coordinate. Then, for $1 \leq i \leq k$, $\pi_i \circ \phi = f^i$. Consequently, $f^i \circ \phi^{-1} = \pi_i$. Thus, if we set $\tilde{f}^i := f^i \circ \phi^{-1}$, for $k + 1 \leq i \leq m$, then

$$f \circ \phi^{-1}(x) = (x^1, \ldots, x^k, \tilde{f}^{k+1}(x), \ldots, \tilde{f}^m(x))$$

for all $x \in U$. Next note that

$$D(f \circ \phi^{-1})(o) = \begin{pmatrix} I_k & 0 \\ \partial(\tilde{f}^{k+1}, \ldots, \tilde{f}^m)(o) \\ (x^{k+1}, \ldots, x^n) \end{pmatrix}.$$ 

On the other hand, $D(f \circ \phi^{-1})(o) = D(f)(p) \circ D(\phi^{-1})(o)$. Thus

$$\text{rank}(D(f \circ \phi^{-1})(o)) = \text{rank}(D(f)(p)) = k,$$

because $D(\phi^{-1}) = D(\phi)^{-1}$ is nonsingular. The last two equalities imply that

$$\frac{\partial(\tilde{f}^{k+1}, \ldots, \tilde{f}^m)}{(x^{k+1}, \ldots, x^n)}(o) = 0,$$

where 0 here denotes the matrix all of whose entries is zero. So we conclude that the functions $\tilde{f}^{k+1}, \ldots, \tilde{f}^m$ do not depend on $x^{k+1}, \ldots, x^n$. In particular, if $V$ is a small neighborhood of $o$ in $\mathbb{R}^m$, then the mapping $T: V \to \mathbb{R}^m$ given by

$$T(y) := (y^1, \ldots, y^k, y^{k+1} + f^{k+1}(y^1, \ldots, y^k), \ldots, y^m + f^m(y^1, \ldots, y^k))$$

is well defined. Now note that

$$DT(o) = \begin{pmatrix} I_k & 0 \\ 0 & I_{m-k} \end{pmatrix}.$$ 

Thus, by the inverse function theorem, $\psi := T^{-1}$ is well defined on an open neighborhood of $o$ in $\mathbb{R}^m$. Finally note that

$$\psi \circ f \circ \phi^{-1}(x) = \psi(x^1, \ldots, x^k, \tilde{f}^{k+1}(x), \ldots, \tilde{f}^m(x))$$

$$= \psi \circ T(x^1, \ldots, x^k, 0, \ldots, 0)$$

$$= (x^1, \ldots, x^k, 0, \ldots, 0),$$

as desired.

\textbf{Exercise 13.} Show that there exists no $C^1$ function $f: \mathbb{R}^2 \to \mathbb{R}$ which is one-to-one.