1.17 The Frenet-Serret Frame and Torsion

Recall that if \( \alpha : I \rightarrow \mathbb{R}^n \) is a unit speed curve, then the unit tangent vector is defined as

\[
T(t) := \alpha'(t).
\]

Further, if \( \kappa(t) = \|T'(t)\| \neq 0 \), we may define the principal normal as

\[
N(t) := \frac{T'(t)}{\kappa(t)}.
\]

As we saw earlier, in \( \mathbb{R}^2 \), \( \{T, N\} \) form a moving frame whose derivatives may be expressed in terms of \( \{T, N\} \) itself. In \( \mathbb{R}^3 \), however, we need a third vector to form a frame. This is achieved by defining the binormal as

\[
B(t) := T(t) \times N(t).
\]

Then similar to the computations we did in finding the derivatives of \( \{T, N\} \), it is easily shown that

\[
\begin{pmatrix}
T(t) \\
N(t) \\
B(t)
\end{pmatrix}' = \begin{pmatrix}
0 & \kappa(t) & 0 \\
-\kappa(t) & 0 & \tau(t) \\
0 & -\tau(t) & 0
\end{pmatrix}
\begin{pmatrix}
T(t) \\
N(t) \\
B(t)
\end{pmatrix},
\]

where \( \tau \) is the torsion which is defined as

\[
\tau(t) := -\langle B', N \rangle.
\]

Note that torsion is well defined only when \( \kappa \neq 0 \), so that \( N \) is defined. Torsion is a measure of how much a space curve deviates from lying in a plane:

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Exercise 1. Show that if the torsion of a curve \( \alpha : I \to \mathbb{R}^3 \) is zero everywhere then it lies in a plane. (Hint: We need to check that there exist a point \( p \) and a (fixed) vector \( v \) in \( \mathbb{R}^3 \) such that \( \langle \alpha(t) - p, v \rangle = 0 \). Let \( v = B \), and \( p \) be any point of the curve.)

Exercise 2. Computer the curvature and torsion of the circular helix 

\[
(r \cos t, r \sin t, ht)
\]

where \( r \) and \( h \) are constants. How does changing the values of \( r \) and \( h \) effect the curvature and torsion.

1.18 Curves of Constant Curvature and Torsion

The above exercise shows that the curvature and torsion of a circular helix are constant. The converse is also true

Theorem 3. The only curve \( \alpha : I \to \mathbb{R}^3 \) whose curvature and torsion are nonzero constants is the circular helix.

The rest of this section develops a number of exercises which leads to the proof of the above theorem

Exercise 4. Show that \( \alpha : I \to \mathbb{R}^3 \) is a circular helix (up to rigid motion) provided that there exists a vector \( v \) in \( \mathbb{R}^3 \) such that 

\[
\langle T, v \rangle = \text{const},
\]

and the projection of \( \alpha \) into a plane orthogonal to \( v \) is a circle.

So first we need to show that when \( \kappa \) and \( \tau \) are constants, \( v \) of the above exercise can be found. We do this with the aid of the Frenet-Serret frame. Since \( \{T, N, B\} \) is an orthonormal frame, we may arite

\[
v = a(t)T(t) + b(t)N(t) + c(t)B(t).
\]

Next we need to find \( a, b \) and \( c \) subject to the conditions that \( v \) is a constant vector, i.e., \( v' = 0 \), and that \( \langle T, v \rangle = \text{const} \). The latter implies that 

\[
a = \text{const}
\]

because \( \langle T, v \rangle = a \). In particular, we may set \( a = 1 \).
Exercise 5. By setting \( v' = 0 \) show that

\[
v = T + \frac{\kappa}{\tau} B,
\]

and check that \( v \) is the desired vector, i.e. \( \langle T, v \rangle = \text{const} \) and \( v' = 0 \).

So to complete the proof of the theorem, only the following remains:

Exercise 6. Show that the projection of \( \alpha \) into a plane orthogonal to \( v \), i.e.,

\[
\bar{\alpha}(t) := \alpha(t) - \langle \alpha(t), v \rangle \frac{v}{\|v\|^2}
\]

is a circle. (Hint: Compute the curvature of \( \bar{\alpha} \).)

1.19 Intrinsic Characterization of Spherical Curves

In this section we derive a characterization in terms of curvature and torsion for unit speed curves which lie on a sphere. Suppose \( \alpha: I \to \mathbb{R}^3 \) lies on a sphere of radius \( r \). Then there exists a point \( p \) in \( \mathbb{R}^3 \) (the center of the sphere) such that

\[
\|\alpha(t) - p\| = r.
\]

Thus differentiation yields

\[
\langle T(t), \alpha(t) - p \rangle = 0.
\]

Differentiating again we obtain:

\[
\langle T'(t), \alpha(t) - p \rangle + 1 = 0.
\]

The above expression shows that \( \kappa(t) \neq 0 \). Consequently \( N \) is well defined, and we may rewrite the above expression as

\[
\kappa(t)\langle N(t), \alpha(t) - p \rangle + 1 = 0.
\]

Differentiating for the third time yields

\[
\kappa'(t)\langle N(t), \alpha(t) - p \rangle + \kappa(t)\langle -\kappa(t)T(t) + \tau(t)B(t), \alpha(t) - p \rangle = 0,
\]

which using the previous expressions above we may rewrite as

\[
- \frac{\kappa'(t)}{\kappa(t)} + \kappa(t)\tau(t)\langle B(t), \alpha(t) - p \rangle = 0.
\]
Next note that, since \( \{T, N, B\} \) is orthonormal,
\[
\begin{align*}
r^2 &= \|\alpha(t) - p\|^2 \\
&= \langle \alpha(t) - p, T(t) \rangle^2 + \langle \alpha(t) - p, N(t) \rangle^2 + \langle \alpha(t) - p, B(t) \rangle^2 \\
&= 0 + \frac{1}{\kappa^2(t)} + \langle \alpha(t) - p, B(t) \rangle^2.
\end{align*}
\]

Thus, combining the previous two calculations, we obtain:
\[
\frac{-\kappa'(t)}{\kappa(t)} + \kappa(t)\tau(t)\sqrt{r^2 - \frac{1}{\kappa^2(t)}} = 0.
\]

**Exercise 7.** Check the converse, that is supposing that the curvature and torsion of some curve satisfies the above expression, verify whether the curve has to lie on a sphere of radius \( r \).

To do the above exercise, we need to first find out where the center \( p \) of the sphere could lie. To this end we start by writing
\[
p = \alpha(t) + a(t)T(t) + b(t)N(t) + c(t)B(t),
\]
and try to find \( a(t), b(t) \) and \( c(t) \) so that \( p' = (0, 0, 0) \), and \( \|\alpha(t) - p\| = r \). To make things easier, we may note that \( \alpha(t) = 0 \) (why?). Then we just need to find \( b(t) \) and \( c(t) \) subject to the two constraints mentioned above. We need to verify whether this is possible, when \( \kappa \) and \( \tau \) satisfy the above expression.

### 1.20 The Local Canonical form

In this section we show that all \( C^3 \) curve in \( \mathbb{R}^3 \) essentially look the same in the neighborhood of any point which has nonvanishing curvature and a given sign for torsion.

Let \( \alpha: (-\epsilon, \epsilon) \to \mathbb{R}^3 \) be a \( C^3 \) curve. By Taylor’s theorem
\[
\alpha(t) = \alpha(0) + \alpha'(0)t + \frac{1}{2}\alpha''(0)t^2 + \frac{1}{6}\alpha'''(0)t^3 + R(t)
\]
where \( \lim_{t \to 0} |R(t)|/t^3 = 0 \), i.e., for \( t \) small, the remainder term \( R(t) \) is negligible. Now suppose that \( \alpha \) has unit speed. Then

\[
\begin{align*}
\alpha' &= T \\
\alpha'' &= T' = \kappa N \\
\alpha''' &= (\kappa N)' = \kappa' N + \kappa (-\kappa T + \tau B) = -\kappa^2 T + \kappa' N + \tau B.
\end{align*}
\]

So we have

\[
\alpha(t) = \alpha(0) + T_0 t + \frac{\kappa_0 N_0 t^2}{2} + \frac{(-\kappa_0^2 T_0 + \kappa_0' N_0 + \tau_0 B_0)t^3}{6} + R(t) = \alpha(0) + (t - \frac{\kappa_0^2}{6} t^3)T_0 + (\frac{\kappa_0}{2} t^2 + \frac{\kappa_0'}{6} t^3)N_0 + (\frac{-\kappa_0 \tau_0}{6} t^3)B_0 + R(t)
\]

Now if, after a rigid motion, we suppose that \( \alpha(0) = (0, 0, 0) \), \( T = (1, 0, 0) \), \( N = (0, 1, 0) \), and \( B = (0, 0, 1) \), then we have

\[
\alpha(t) = \left( t - \frac{\kappa_0^2}{6} t^3 + R_x, \frac{\kappa_0}{2} t^2 + \frac{\kappa_0'}{6} t^3 + R_y, \frac{-\kappa_0 \tau_0}{6} t^3 + R_z \right),
\]

where \((R_x, R_y, R_z) = R\). It follows then that when \( t \) is small

\[
\alpha(t) \approx \left( t, \frac{\kappa_0}{2} t^2, \frac{-\kappa_0 \tau_0}{6} t^3 \right).
\]

Thus, up to third order of differentiation, any curve with nonvanishing curvature in space may be approximated by a cubic curve. Also note that the above approximation shows that any planar curve with nonvanishing curvature locally looks like a parabola.

**Exercise 8.** Show that the curvature of a space curve \( \alpha \) at a point \( t_0 \) with nonvanishing curvature is the same as the curvature of the projection of \( \alpha \) into the osculating plane at time \( t_0 \). (The osculating plane is the plane generated by \( T \) and \( N \).)