1. Let
\[
A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 5 & 1 \end{bmatrix}.
\]
Find a permutation matrix \( P \), a lower triangular matrix \( L \) with ones along the main diagonal and an upper triangular matrix \( U \) such that
\[ PA = LU. \]

2. Let
\[
V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x + y + z = 0 \right\}.
\]
(a) Show that \( V \) is a vector subspace of \( \mathbb{R}^3 \) and find its dimension.
(b) Let
\[
a = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.
\]
Show that \( a, b, c \) are vectors in \( V \) and use these vectors to find an orthogonal basis for \( V \).
(c) Let \( A \) be the three by three matrix \([a \ b \ c]\). Find an orthogonal matrix \( Q \) and an upper triangular matrix \( R \) such that \( A = QR \).

There are six more problems on the next two pages
3. Let
\[
A = \begin{bmatrix}
1 & -1 & 2 \\
2 & -1 & 1 \\
-1 & 0 & 1 \\
\end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix}
-1 \\
4 \\
-1 \\
\end{bmatrix}.
\]
(a) Find the dimensions of the row space and null space of $A$.
(b) Find a basis for the column space of $A$.
(c) Is the least squares solution to the system $Ax = b$ unique? Why or why not?

4. Let
\[
A = \begin{bmatrix}
1 & 0 & 2 \\
1 & 1 & 4 \\
\end{bmatrix}.
\]
(a) Find a basis for the orthogonal complement of the row space of $A$.
(b) Let
\[
\mathbf{x} = \begin{bmatrix}
1 \\
1 \\
1 \\
\end{bmatrix}.
\]
Write $\mathbf{x}$ as the sum of an element in the row space of $A$ and an element in its orthogonal complement.

5. Let $A$ be a rectangular matrix. Prove that $A$ and $A^tA$ have the same null space.

6. Find two eigenvalues of the matrix
\[
A = \begin{bmatrix}
.2 & .4 & .3 \\
.4 & .2 & .3 \\
.4 & .4 & .4 \\
\end{bmatrix}.
\]
\textbf{(Hint: sum along columns.)}

There are two more problems on the next page
7. Let
\[ A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} . \]

(a) Find the determinant, the trace, and the characteristic polynomial of \( A \).

(b) Find all the eigenvalues and eigenvectors of \( A \).

(c) Find the singular value decomposition and the pseudoinverse \( A^+ \) of \( A \).

(d) Find the least squares solution of minimal norm to the system
\[ Ax = \begin{bmatrix} 1 \\ 1 \end{bmatrix} . \]

(e) Find the maximum value of
\[ f(x, y) = \frac{x^2 - 2xy + y^2}{x^2 + y^2} . \]

At what vectors \( \begin{bmatrix} x \\ y \end{bmatrix} \) is this maximum achieved and why?

8. Let
\[ A = \begin{bmatrix} 1 & a \\ a & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & a & 0 \\ a & 1 & 0 \\ 0 & 0 & b \end{bmatrix} . \]

(a) For precisely what choices of \( a \) is \( A \) positive definite? Justify your answer.

(b) For precisely what choices of \( a \) and \( b \) is \( B \) positive definite? Justify your answer.

(c) Choose a value of \( b \) which makes \( B \) indefinite. Justify your answer.

(d) Choose a value of \( a \) which makes \( B \) indefinite. Justify your answer.

(e) Suppose \( a = 0 \). Precisely when is \( B \) positive semidefinite? Justify your answer.
Solutions

1.

\[
\begin{bmatrix}
1 & 0 & 0 \\
-1 & 1 & 0 \\
-2 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 \\
0 & 0 & 2 \\
0 & 3 & -1
\end{bmatrix}
= 
\begin{bmatrix}
1 & 0 & 0 \\
-2 & 1 & 0 \\
-1 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 \\
0 & 3 & -1 \\
0 & 0 & 2
\end{bmatrix}
\]

Therefore,

\[
\begin{bmatrix}
1 & 0 & 0 \\
2 & 1 & 0 \\
1 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 \\
0 & 3 & -1 \\
0 & 0 & 2
\end{bmatrix}
= 
\begin{bmatrix}
1 & 1 & 1 \\
0 & 3 & -1 \\
0 & 0 & 2
\end{bmatrix}
\]

2.

(a) Let \( \alpha \in \mathbb{R} \) and \( x = [x_1 \ x_2 \ x_3]^t \), \( y = [y_1 \ y_2 \ y_3]^t \) two elements in \( V \). Since \( 0 = \alpha \cdot 0 = \alpha(x_1 + x_2 + x_3) = (\alpha x_1) + (\alpha x_2) + (\alpha x_3) \) this means that \( \alpha x \in V \). Similarly, \( 0 = 0 + 0 = (x_1 + x_2 + x_3) + (y_1 + y_2 + y_3) = (x_1 + y_1) + (x_2 + y_2) + (x_3 + y_3); \) therefore, \( x + y \in V \). Since this held for any scalar and any two members of \( V \), it follows that \( V \) is a vector subspace of \( \mathbb{R}^3 \).

(b) It is clear that the sum of the components of \( a, b \) and \( c \) equals zero, so these vectors belong to \( V \). To find an orthogonal basis for \( V \) out of these three vectors we apply the Gram Schimdt algorithm. Note that the basis elements need not be unit length. So, let

\[
v_1 = a \quad .
\]

\[
v_2 = b - \frac{b \cdot v_1}{\|v_1\|^2} v_1
\]

\[
= \begin{bmatrix}
0 \\
1 \\
-1
\end{bmatrix} - \frac{1}{2} \begin{bmatrix}
1 \\
-1 \\
0
\end{bmatrix} = \begin{bmatrix}
1/2 \\
1/2 \\
-1
\end{bmatrix} \quad .
\]

\( v_1 \) and \( v_2 \) constitute a basis for \( V \). Indeed, \( v_1 \) and \( v_2 \) belong to \( V \), which is a two—dimensional vector subspace \( (x = -y - z \) means that \( x \) is a pivotal variable while \( y \) and \( z \) are free; therefore, \( V \)
has dimension equal to two).

(c)

\[ Q = \begin{bmatrix} v_1 & v_2 \\ \|v_1\| & \|v_2\| \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{6} \\ 0 & -\sqrt{2/3} \end{bmatrix}. \]

\[ R = Q^t A = \begin{bmatrix} \sqrt{2} & -1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & \sqrt{3/2} & \sqrt{3/2} \end{bmatrix}. \]

3.

(a) Row reducing the augmented matrix of the system \( Ax = b \) we obtain:

\[
\begin{bmatrix} A & b \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 2 & -1 \\ 0 & 1 & -3 & 6 \\ 0 & -1 & 3 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 2 & -1 \\ 0 & 1 & -3 & 6 \\ 0 & 0 & 0 & 4 \end{bmatrix}.
\]

The system has no solutions in the ordinary sense (i.e., \( b \notin R(A) \)). From above we see that the first two columns of \( A \) are pivotal; therefore, the dimension of its row space is two (which is also the dimension of its column space; i.e., its rank). The dimension of its null space is equal to the number of columns minus its rank; therefore, the null space of \( A \) has dimension one.

(b) The pivotal columns of \( A \) are a basis for its column space. In this case, the pivotal columns are the first two.

(c) The least squares solution is not unique in the sense that the normal equations, \( A^tAx = A^tb \), have infinitely many solutions (\( A^tA \) is not invertible).

4.

(a) \( (R(A^t))^\perp = N(A) \) so, we are asked to find a basis for the null space of \( A \).

\[ A \rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}; \]

therefore, the system \( Ax = 0 \) is equivalent to

\[
x_1 = -2x_3, \\
x_2 = -2x_3.
\]
Setting $x_3 = 1$ we obtain that $n = [-2, -2, 1]^t$ is a basis vector for the null space of $A$.

(b)

$$x = \left( x - \frac{x \cdot n}{\|n\|^2} n \right) + \frac{x \cdot n}{\|n\|^2} n$$

$$= \left( x - \frac{-3}{9} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right) + \frac{-3}{9} \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 \\ 1/3 \\ 4/3 \end{bmatrix} + \begin{bmatrix} 2/3 \\ 2/3 \\ -1/3 \end{bmatrix}.$$

5. Let $v \in N(A)$ then $A^t Av = A^t(Av) = A^t 0 = 0$; therefore, $N(A) \subset N(A^tA)$. In the other direction, suppose that $u \in N(A^tA)$ so that $A^tAu = 0$. Then $0 = u^t0 = u^tA^tAu = (Au)^t(Au) = \|Au\|^2$, which holds iff $Au = 0$; i.e., iff $u \in N(A)$, thus $N(A^tA) \subset N(A)$. So that $N(A^tA) = N(A)$.

6. The components on each one of the columns of $A$ add up to one; furthermore, they are positive; therefore, $A$ is a Markov matrix and thus one of its eigenvalues must be one, $s_1 = 1$. Aside, observe that the columns of $A$ are ld; the third column of $A$ is half of the sum of the first two; therefore, $s_2 = 0$ is also an eigenvalue of $A$. Moreover, when you subtract the second column from the first you get $[-2, 2, 0]^t = -2[1, -1, 0]^t$. The operation of subtracting the second column of $A$ from the first is equivalent to multiplying $A$ by the vector $[1, -1, 0]^t$ so, this vector is an eigenvector and thus $s_3 = -2$ is its eigenvalue. You can also get the third eigenvalue from knowing that the sum of all the eigenvalues is equal to the trace of $A$ which is equal to .8.

7.

(a) the $\det(A) = 0$ (columns are ld –the first is the negative of the second). $\text{tr}(A) = 2, p(s) = \det(A - sI) = (1 - s)^2 - 1 = s(s - 2)$.

(b) $s_1 = 2$ and $s_2 = 0$ are the eigenvalues of $A$. $A$ is symmetric; therefore, its eigenvectors can be chosen to be perpendicular. Inspection of the matrix $A$ gives us the eigenvectors corresponding to $s_1$ and $s_2$: $v_1 = [1, -1]^t, v_2 = [1, 1]^t$; respectively.
(c) Since $A$ is symmetric, its svd is simply its diagonalization using its eigenvectors. Let $u_1 = \frac{v_1}{||v_1||}$ and $u_2 = \frac{v_2}{||v_2||}$ be unit eigenvectors and define $Q = [v_1 \ v_2]$, then

$$AQ = QA$$

$$A = QAQ^t$$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$.

Once the svd is known, the pseudoinverse can be found as follows:

$$A^+ = Q \begin{bmatrix} 1/2 & 0 \\ 0 & 0 \end{bmatrix} Q^t$$

$$= \begin{bmatrix} 1/2\sqrt{2} & 0 \\ -1/2\sqrt{2} & 0 \end{bmatrix} Q^t$$

$$= \begin{bmatrix} 1/4 & -1/4 \\ -1/4 & 1/4 \end{bmatrix} = \frac{1}{4}A$$.

(d) The minimum length least squares solution is given by

$$x^+ = A^+ \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$$.

(e) Let $x = [x, y]^t$, then observe that

$$f(x, y) = \frac{x^tA}{||x||}x$$,

where $A$ is the matrix given at the beginning. $f$ is then a (quadratic) function defined on the unit circle, it's maximum can be found using elementary calculus; however, the svd says that its maximum is achieved in the direction of the eigenvector with the largest eigenvalue, which in this case is $u_1$; therefore,

$$\max\{f\} = f(u_1) = u_1^tAu_1 = 2$$.

8. Observe that $A$ and $B$ are symmetric.

(a) A sufficient condition to guarantee positive definiteness of $A$ and $B$ is that their eigenvalues be all positive.

$$p(s) = \det(A - sI) = (1 - s)(4 - s) - a^2 = s^2 - 5s + 4 - a^2$$

$$= \left(s - \frac{5}{2}\right)^2 - \frac{25}{4} + 4 - a^2 = \left(s - \frac{5}{2}\right)^2 - \frac{9}{4} - a^2$$

$$= \left(s - \frac{5}{2}\right)^2 - \frac{9 + 4a^2}{4}$$.
The eigenvalues of $A$ are thus

$$s_{\pm} = \frac{1}{2}(5 \pm \sqrt{9 + 4a^2}) .$$

$s_+ > 0$. $s_- > 0$ iff $5 > \sqrt{9 + 4a^2}$, i.e., $25 > 9 + 4a^2 \Rightarrow 4 > a^2 \Rightarrow -2 < a < 2$. Thus $A$ is positive definite iff $a \in (-2, 2)$.

(b) Similarly,

$$q(s) = \det(B - sI) = (b - s)\det(A - sI) = (b - s)p(s) ,$$

the eigenvalues of $B$ are then $b$, $s_-$, $s_+$, so that $B$ is positive definite iff $b > 0$ and $a \in (-2, 2)$.

(c) Choosing any $b < 0$ will make $B$ indefinite, say $b = -1$. The reason for this is that $s_+$ is always positive so, if $b$ is negative there will be positive and negative eigenvalues which guarantees that the quadratic form $x^tBx$ will take negative and positive values inside of any neighborhood of the origin.

(d) Similarly, if $|a| > 2$, say $a = 3$, then $s_+ < 0$, but $s_+$ is always positive so, also in this case, $A$ will have positive and negative eigenvalues, making the quadratic form of part (c) (and therefore $B$) indefinite.

(e) If $a = 0$ then $s_+$ and $s_-$ are both positive. In order to make $B$ positive semidefinite we need to guarantee that the quadratic form $x^tBx$ will vanish at points inside of any neighborhood of the origin other than the origin itself. This is accomplished by setting $b = 0$; in which case, the quadratic form vanishes along the line in the direction of the vector $[0 \ 0 \ 1]^t$. Obviously, this line passes through the origin and thus it intersects every neighborhood of the origin, making the aforementioned quadratic form (and thus $B$) indefinite.