Instructions: Work absolutely on your own, without notes or text. Your grade will be based Problem 1 and on the best two of the other three problems.

You may do, or redo, the fourth problem for homework next week.

1. (S-S 2.6 # 15.) Suppose \( f \) is a nonvanishing continuous function on the closed unit disc and holomorphic on the open unit disc. Prove that if

\[
|f(z)| = 1 \quad \text{whenever} \quad |z| = 1,
\]

then \( f \) is constant.

2. Evaluate

\[
\int_{\gamma} \frac{z^2 - 1}{z(z^2 + 9)} \, dz,
\]

where \( \gamma = re^{it}, 0 \leq t \leq 2\pi, \) for all possible values of \( r, 0 < r < 3 \) and \( 3 < r < \infty. \)

3. Each of the following functions has an isolated singularity at \( z=0. \) Determine its nature. If it is a removable singularity, define \( f(0) \) so \( f \) is holomorphic there. If it is a pole, find the principal part of \( f. \) If it is an essential singularity, find the image under \( f \) of \( \{z : 0 < |z| < \delta\} \) for arbitrarily small \( \delta. \)

a. \( \frac{\cos(2z) - 1}{z}. \)

b. \( z \cos \left( \frac{2}{z} \right) \)

c. \( (\cos(2z) - 1)^{-1} \)

4. Suppose that \( f(z) \) is a continuous mapping on \( \mathbb{C} \) that preserves all distances. Is it true that \( f(z) \) is a composition of translations \( z \rightarrow z + a, \) rotations \( z \rightarrow wz \) for some \( |w| = 1, \) and possibly complex conjugation? Prove or disprove.