1. If \( r \in \mathbb{R} \) is irrational and \( q \in \mathbb{R}, \; q \neq 0 \) is rational, show from the definitions of (ir)rational numbers and the properties of the integers, that \( r + q \) and \( rq \) are irrational.

2. Let \( a \in \mathbb{R} \) with \( a > -1 \). Prove that \( (1 + a)^n \geq 1 + na \) for all \( n \in \mathbb{N} \).

3. Show that if \( a \leq x \leq b \) and \( a \leq y \leq b \) for \( a, b, x, y \in \mathbb{R} \), then \( |x - y| \leq b - a \). Also give a geometric interpretation for this.

4. Let \( X \subseteq Y \subseteq \mathbb{R} \) be bounded, non-empty sets. Prove that \( \inf Y \leq \inf X \leq \sup X \leq \sup Y \).

5. Consider nonempty sets \( X, Y \subseteq \mathbb{R} \) with \( X \cap Y = \emptyset \) and \( X \cup Y = \mathbb{R} \). Suppose that \( x < y \) for all \( x \in X, \; y \in Y \). Prove there exists a unique \( \alpha \in \mathbb{R} \) such that \( x \leq \alpha \) for all \( x \in X \) and \( \alpha \leq y \) for all \( y \in Y \).

6. Consider a set \( X \neq \emptyset \). Let \( f \) and \( g \) be functions \( f : X \to \mathbb{R} \) and \( g : X \to \mathbb{R} \) both having bounded ranges. Show that

\[
\inf\{f(x) \mid x \in X\} + \inf\{g(x) \mid x \in X\} \leq \inf\{f(x) + g(x) \mid x \in X\} \leq \inf\{f(x) \mid x \in X\} + \sup\{g(x) \mid x \in X\} \leq \sup\{f(x) + g(x) \mid x \in X\} \leq \sup\{f(x) \mid x \in X\} + \sup\{g(x) \mid x \in X\}
\]

Also, give examples to show that each inequality is strict.