1. Determine whether each of the following functions \(d(x, y)\) for \(x, y \in \mathbb{R}\) is a metric on \(\mathbb{R}\). Justify your answer.
   
   (a) \(d(x, y) = (x - y)^2\).
   (b) \(d(x, y) = \sqrt{|x - y|}\).
   (c) \(d(x, y) = |x^2 - y^2|\).
   (d) \(d(x, y) = |x - 2y|\).
   (e) \(d(x, y) = \frac{|x - y|}{1 + |x - y|}\).

2. Let \(V\) be a vector space over \(\mathbb{R}\) with a norm denoted \(\|x\|\) for \(x \in V\). Consider the function \(d(x, y) = \|x - y\|\) for \(x, y \in V\).
   
   (a) Prove that \(d(x, y)\) is a metric on \(V\).
   (b) Prove that \(d(x, y)\) is translation invariant, that is \(d(x, y) = d(x + z, y + z)\) for all \(z \in V\).
   (c) Prove that \(d(x, y)\) is homogeneous, that is \(d(ax, ay) = |a|d(x, y)\) for all \(a \in \mathbb{R}\).

3. Let \(V\) be a vector space over \(\mathbb{R}\) with a metric \(d(x, y)\) for \(x, y \in V\). Consider the function \(\|x\| = d(x, 0)\) for \(x \in V\) where 0 denotes the zero vector in \(V\).
   
   (a) Assume that \(d(x, y)\) is both translation invariant and homogeneous. (See problem 2.) Prove that \(\|x\|\) is a norm on \(V\).
   (b) What fails if \(d(x, y)\) is not translation invariant?
   (c) What fails if \(d(x, y)\) is not homogeneous?

4. Consider the following norms on \(\mathbb{R}^n\) where \(x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n\): \(\|x\|_1 = \sum_{i=1}^{n}|x_i|\), \(\|x\|_2 = (\sum_{i=1}^{n} x_i^2)^{1/2}\), \(\|x\|_\infty = \max\{|x_i| \mid 1 \leq i \leq n\}\).
   
   (a) Show there exist \(a, b \in \mathbb{R}\) with \(a, b > 0\) such that \(a\|x\|_1 \leq \|x\|_2 \leq b\|x\|_1\) for all \(x \in \mathbb{R}^n\). What is the largest \(a\) and smallest \(b\) with this property?
   (b) Show there exist \(a, b \in \mathbb{R}\) with \(a, b > 0\) such that \(a\|x\|_1 \leq \|x\|_\infty \leq b\|x\|_1\) for all \(x \in \mathbb{R}^n\). What is the largest \(a\) and smallest \(b\) with this property?

5. A norm \(\|x\|\) for \(x \in \mathbb{R}^n\) satisfies the parallelogram identity if (and only if) the relation \(\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)\) holds for all \(x, y \in \mathbb{R}^n\).
   
   (a) Show that \(\|x\|_2\) satisfies the parallelogram identity.
   (b) Illustrate this in \(\mathbb{R}^2\) and explain what it means geometrically.
   (c) Show that \(\|x\|_1\) and \(\|x\|_\infty\) do not satisfy the parallelogram identity.