1. Let \((E, d)\) and \((E', d')\) be metric spaces with \(f : E \to E'\) a function. Suppose \(f\) is continuous.

   (a) Prove that if \(T \subseteq f(E)\) is open in \(f(E)\), then \(f^{-1}(T)\) is open in \(E\).

   (b) Suppose \(E\) is compact. Prove if \(S \subseteq E\) is closed, then \(f(S) \subseteq E'\) is closed.

   (c) Suppose \(E\) is compact and \(f\) is a bijection. Prove that \(f^{-1} : E' \to E\) is continuous.

2. Let \((E, d)\) be a metric space with points \(p, q \in E\). We say there is a path from \(p\) to \(q\) in \(E\) if there is a continuous function \(f : [0, 1] \to E\) such that \(f(0) = p\) and \(f(1) = q\). Then \((E, d)\) is defined to be path connected if there is some path between every two points \(p, q \in E\). A subset \(S \subseteq E\) is path connected if it is path connected as a subspace.

   (a) Prove that a path connected metric space is connected.

   (b) Prove that any connected open subset of \(E^n\) is path connected.

3. Let \((E, d)\) be a metric space and \(S \subseteq E\). Suppose that \(S\) has the property that each point of \(c(S)\) is a cluster point of \(S\). (Such a set \(S\) is called dense in \(E\).) Let \(E'\) be a complete metric space and \(f : S \to E'\) a uniformly continuous function.

   (a) Prove that \(f\) can be extended to a continuous function \(g\) from \(E\) into \(E'\).

   (b) Prove that if \(h\) is another extension of \(f\) from \(E\) into \(E'\), then \(g = h\). (This shows the extension is unique.)

   (c) Prove that \(g\) is uniformly continuous.

4. Let \((E, d)\) be a nonempty compact metric space. Prove that \(\{d(p, q) \mid p, q \in E\}\) has a greatest element. (This shows existence of \(\max\{d(p, q) \mid p, q \in E\}\), which is called the diameter of \(E\).)

   Hint: are there sequences \(p_n\) and \(q_n\) such that \(\lim_{n \to \infty} d(p_n, q_n) = \sup\{d(p, q) \mid p, q \in E\}\), if that exists?