department of a university where the multiplicities are the number of pieces of each type needed, and $B$ is the analogous multiset for a second department of the university. For instance, $A$ could be the multiset \{107 \cdot \text{personal computers}, 44 \cdot \text{routers}, 6 \cdot \text{servers}\} and $B$ could be the multiset \{14 \cdot \text{personal computers}, 6 \cdot \text{routers}, 2 \cdot \text{mainframes}\}.

a) What combination of $A$ and $B$ represents the equipment the university should buy assuming both departments use the same equipment?

b) What combination of $A$ and $B$ represents the equipment that will be used by both departments if both departments use the same equipment?

c) What combination of $A$ and $B$ represents the equipment that the second department uses, but the first department does not, if both departments use the same equipment?

d) What combination of $A$ and $B$ represents the equipment that the university should purchase if the departments do not share equipment?

**Fuzzy sets** are used in artificial intelligence. Each element in the universal set $U$ has a degree of membership, which is a real number between 0 and 1 (including 0 and 1), in a fuzzy set $S$. The fuzzy set $S$ is denoted by listing the elements with their degrees of membership (elements with 0 degree of membership are not listed). For instance, we write \{0.6 \text{ Alice}, 0.9 \text{ Brian}, 0.4 \text{ Fred}, 0.1 \text{ Oscar}, 0.5 \text{ Rita}\} for the set $F$ (of famous people) to indicate that Alice has a 0.6 degree of membership in $F$, Brian has a 0.9 degree of membership in $F$, Fred has a 0.4 degree of membership in $F$, Oscar has a 0.1 degree of membership in $F$, and Rita has a 0.5 degree of membership in $F$ (so that Brian is the most famous and Oscar is the least famous of these people). Also suppose that $R$ is the set of rich people with $R = \{0.4 \text{ Alice}, 0.8 \text{ Brian}, 0.2 \text{ Fred}, 0.9 \text{ Oscar}, 0.7 \text{ Rita}\}$.

49. The complement of a fuzzy set $S$ is the set $\overline{S}$, with the degree of the membership of an element in $\overline{S}$ equal to 1 minus the degree of membership of this element in $S$. Find $\overline{F}$ (the fuzzy set of people who are not famous) and $\overline{R}$ (the fuzzy set of people who are not rich).

50. The union of two fuzzy sets $S$ and $T$ is the fuzzy set $S \cup T$, where the degree of membership of an element in $S \cup T$ is the maximum of the degrees of membership of this element in $S$ and in $T$. Find the fuzzy set $F \cup R$ of rich or famous people.

51. The intersection of two fuzzy sets $S$ and $T$ is the fuzzy set $S \cap T$, where the degree of membership of an element in $S \cap T$ is the minimum of the degrees of membership of this element in $S$ and in $T$. Find the fuzzy set $F \cap R$ of rich and famous people.

### 1.6 Functions

**INTRODUCTION**

In many instances we assign to each element of a set a particular element of a second set (which may be the same as the first). For example, suppose that each student in a discrete mathematics class is assigned a letter grade from the set \{A, B, C, D, F\}. And suppose that the grades are $A$ for Adams, $C$ for Chou, $B$ for Goodfriend, $A$ for Rodriguez, and $F$ for Stevens. This assignment of grades is illustrated in Figure 1.

![Assignment of Grades in a Discrete Mathematics Class.](image-url)
This assignment is an example of a function. The concept of a function is extremely important in discrete mathematics. Functions are used in the definition of such discrete structures as sequences and strings. Functions are also used to represent how long it takes a computer to solve problems of a given size. Recursive functions, which are functions defined in terms of themselves, are used throughout computer science; they will be studied in Chapter 3. This section reviews the basic concepts involving functions needed in discrete mathematics.

**Definition 1.** Let $A$ and $B$ be sets. A function $f$ from $A$ to $B$ is an assignment of exactly one element of $B$ to each element of $A$. We write $f(a) = b$ if $b$ is the unique element of $B$ assigned by the function $f$ to the element $a$ of $A$. If $f$ is a function from $A$ to $B$, we write $f : A \rightarrow B$.

Functions are specified in many different ways. Sometimes we explicitly state the assignments. Often we give a formula, such as $f(x) = x + 1$, to define a function. Other times we use a computer program to specify a function.

**Definition 2.** If $f$ is a function from $A$ to $B$, we say that $A$ is the domain of $f$ and $B$ is the codomain of $f$. If $f(a) = b$, we say that $b$ is the image of $a$ and $a$ is a pre-image of $b$. The range of $f$ is the set of all images of elements of $A$. Also, if $f$ is a function from $A$ to $B$, we say that $f$ maps $A$ to $B$.

Figure 2 represents a function $f$ from $A$ to $B$.

Consider the example that began this section. Let $G$ be the function that assigns a grade to a student in our discrete mathematics class. Note that $G$(Adams) = $A$, for instance. The domain of $G$ is the set {Adams, Chou, Goodfriend, Rodriguez, Stevens}, and the codomain is the set \{A, B, C, D, F\}. The range of $G$ is the set \{A, B, C, F\}, because there are students who are assigned each grade except $D$. Also consider the following examples.

Let $f$ be the function that assigns the last two bits of a bit string of length 2 or greater to that string. Then, the domain of $f$ is the set of all bit strings of length 2 or greater, and both the codomain and range are the set \{00, 01, 10, 11\}.

Let $f$ be the function from $\mathbb{Z}$ to $\mathbb{Z}$ that assigns the square of an integer to this integer. Then, $f(x) = x^2$, where the domain of $f$ is the set of all integers, the codomain of $f$ can be chosen to be the set of all integers, and the range of $f$ is the set of all nonnegative integers that are perfect squares, namely, \{0, 1, 4, 9, \ldots\}. 

![Figure 2 The Function $f$ Maps $A$ to $B$.](image-url)
EXAMPLE 3

(For students familiar with Pascal) The domain and codomain of functions are often specified in programming languages. For instance, the Pascal statement

\[
\text{function floor(x: real): integer}
\]

states that the domain of the floor function is the set of real numbers and its codomain is the set of integers.

Two real-valued functions with the same domain can be added and multiplied.

**DEFINITION 3.** Let \( f_1 \) and \( f_2 \) be functions from \( A \) to \( \mathbb{R} \). Then \( f_1 + f_2 \) and \( f_1f_2 \) are also functions from \( A \) to \( \mathbb{R} \) defined by

\[
(f_1 + f_2)(x) = f_1(x) + f_2(x),
\]

\[
(f_1f_2)(x) = f_1(x)f_2(x).
\]

Note that the functions \( f_1 + f_2 \) and \( f_1f_2 \) have been defined by specifying their values at \( x \) in terms of the values of \( f_1 \) and \( f_2 \) at \( x \).

EXAMPLE 4

Let \( f_1 \) and \( f_2 \) be functions from \( \mathbb{R} \) to \( \mathbb{R} \) such that \( f_1(x) = x^2 \) and \( f_2(x) = x - x^3 \). What are the functions \( f_1 + f_2 \) and \( f_1f_2 \)?

**Solution:** From the definition of the sum and product of functions, it follows that

\[
(f_1 + f_2)(x) = f_1(x) + f_2(x) = x^2 + (x - x^3) = x
\]

and

\[
(f_1f_2)(x) = x^2(x - x^3) = x^3 - x^4.
\]

When \( f \) is a function from a set \( A \) to a set \( B \), the image of a subset of \( A \) can also be defined.

**DEFINITION 4.** Let \( f \) be a function from the set \( A \) to the set \( B \) and let \( S \) be a subset of \( A \). The *image* of \( S \) is the subset of \( B \) that consists of the images of the elements of \( S \). We denote the image of \( S \) by \( f(S) \), so that

\[
f(S) = \{ f(s) \mid s \in S \}.
\]

EXAMPLE 5

Let \( A = \{a, b, c, d, e\} \) and \( B = \{1, 2, 3, 4\} \) with \( f(a) = 2, f(b) = 1, f(c) = 4, f(d) = 1, \) and \( f(e) = 1 \). The image of the subset \( S = \{b, c, d\} \) is the set \( f(S) = \{1, 4\} \).

**ONE-TO-ONE AND ONTO FUNCTIONS**

Some functions have distinct images at distinct members of their domain. These functions are said to be **one-to-one**.
DEFINITION 5. A function \( f \) is said to be one-to-one, or injective, if and only if \( f(x) = f(y) \) implies that \( x = y \) for all \( x \) and \( y \) in the domain of \( f \). A function is said to be an injection if it is one-to-one.

Remark: A function \( f \) is one-to-one if and only if \( f(x) \neq f(y) \) whenever \( x \neq y \). This way of expressing that \( f \) is one-to-one is obtained by taking the contrapositive of the implication in the definition.

We illustrate this concept by giving examples of functions that are one-to-one and other functions that are not one-to-one.

EXAMPLE 6

Determine whether the function \( f \) from \( \{a, b, c, d\} \) to \( \{1, 2, 3, 4, 5\} \) with \( f(a) = 4 \), \( f(b) = 5 \), \( f(c) = 1 \), and \( f(d) = 3 \) is one-to-one.

Solution: The function \( f \) is one-to-one since \( f \) takes on different values at the four elements of its domain. This is illustrated in Figure 3.

EXAMPLE 7

Determine whether the function \( f(x) = x^2 \) from the set of integers to the set of integers is one-to-one.

Solution: The function \( f(x) = x^2 \) is not one-to-one because, for instance, \( f(1) = f(-1) = 1 \), but \( 1 \neq -1 \).

EXAMPLE 8

Determine whether the function \( f(x) = x + 1 \) is one-to-one.

Solution: The function \( f(x) = x + 1 \) is a one-to-one function. To demonstrate this, note that \( x + 1 \neq y + 1 \) when \( x \neq y \).

We now give some conditions that guarantee that a function is one-to-one.

DEFINITION 6. A function \( f \) whose domain and codomain are subsets of the set of real numbers is called strictly increasing if \( f(x) < f(y) \) whenever \( x < y \) and \( x \) and \( y \) are in the domain of \( f \). Similarly, \( f \) is called strictly decreasing if \( f(x) > f(y) \) whenever \( x < y \) and \( x \) and \( y \) are in the domain of \( f \).

These functions correspond to one-to-one functions.

\[
\begin{align*}
& a \rightarrow 1 \\
& b \rightarrow 2 \\
& c \rightarrow 3 \\
& d \rightarrow 4 \\
& \end{align*}
\]

\[\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}\]

\[\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
\end{array}\]

FIGURE 3 A One-to-One Function.
From these definitions, we see that a function that is either strictly increasing or strictly decreasing must be one-to-one.

For some functions the range and the codomain are equal. That is, every member of the codomain is the image of some element of the domain. Functions with this property are called onto functions.

**Definition 7.** A function \( f \) from \( A \) to \( B \) is called onto, or surjective, if and only if for every element \( b \in B \) there is an element \( a \in A \) with \( f(a) = b \). A function \( f \) is called a surjection if it is onto.

We now give examples of onto functions and functions that are not onto.

**Example 9**

Let \( f \) be the function from \( \{a, b, c, d\} \) to \( \{1, 2, 3\} \) defined by \( f(a) = 3 \), \( f(b) = 2 \), \( f(c) = 1 \), and \( f(d) = 3 \). Is \( f \) an onto function?

**Solution:** Since all three elements of the codomain are images of elements in the domain, we see that \( f \) is onto. This is illustrated in Figure 4.

**Example 10**

Is the function \( f(x) = x^2 \) from the set of integers to the set of integers onto?

**Solution:** The function \( f \) is not onto since there is no integer \( x \) with \( x^2 = -1 \), for instance.

**Example 11**

Is the function \( f(x) = x + 1 \) from the set of integers to the set of integers onto?

**Solution:** This function is onto, since for every integer \( y \) there is an integer \( x \) such that \( f(x) = y \). To see this, note that \( f(x) = y \) if and only if \( x + 1 = y \), which holds if and only if \( x = y - 1 \).

**Definition 8.** The function \( f \) is a one-to-one correspondence, or a bijection, if it is both one-to-one and onto.

The following examples illustrate the concept of a bijection.

**Example 12**

Let \( f \) be the function from \( \{a, b, c, d\} \) to \( \{1, 2, 3, 4\} \) with \( f(a) = 4 \), \( f(b) = 2 \), \( f(c) = 1 \), and \( f(d) = 3 \). Is \( f \) a bijection?
1.6 Functions

Solution: The function \( f \) is one-to-one and onto. It is one-to-one since the function takes on distinct values. It is onto since all four elements of the codomain are images of elements in the domain. Hence, \( f \) is a bijection.

Figure 5 displays four functions where the first is one-to-one but not onto, the second is onto but not one-to-one, the third is both one-to-one and onto, and the fourth is neither one-to-one nor onto. The fifth correspondence in Figure 5 is not a function, since it sends an element to two different elements.

Suppose that \( f \) is a function from a set \( A \) to itself. If \( A \) is finite, then \( f \) is one-to-one if and only if it is onto. (This follows from the result in Exercise 58 at the end of this section.) This is not necessarily the case if \( A \) is infinite (as will be shown in Section 1.7).

**EXAMPLE 13**

Let \( A \) be a set. The identit function on \( A \) is the function \( \iota_A : A \to A \) where

\[
\iota_A(x) = x
\]

where \( x \in A \). In other words, the identity function \( \iota_A \) is the function that assigns each element to itself. The function \( \iota_A \) is one-to-one and onto, so that it is a bijection.

**INVERSE FUNCTIONS AND COMPOSITIONS OF FUNCTIONS**

Now consider a one-to-one correspondence \( f \) from the set \( A \) to the set \( B \). Since \( f \) is an onto function, every element of \( B \) is the image of some element in \( A \). Furthermore, because \( f \) is also a one-to-one function, every element of \( B \) is the image of a unique element of \( A \). Consequently, we can define a new function from \( B \) to \( A \) that reverses the correspondence given by \( f \). This leads to the following definition.

**DEFINITION 9.** Let \( f \) be a one-to-one correspondence from the set \( A \) to the set \( B \). The inverse function of \( f \) is the function that assigns to an element \( b \) belonging to \( B \) the unique element \( a \) in \( A \) such that \( f(a) = b \). The inverse function of \( f \) is denoted by \( f^{-1} \). Hence, \( f^{-1}(b) = a \) when \( f(a) = b \).

Figure 6 illustrates the concept of an inverse function.

If a function \( f \) is not a one-to-one correspondence, we cannot define an inverse function of \( f \). When \( f \) is not a one-to-one correspondence, either it is not one-to-one or it is not onto. If \( f \) is not one-to-one, some element \( b \) in the codomain is the image of more than one element in the domain. If \( f \) is not onto, for some element \( b \) in the codomain, no element \( a \) in the domain exists for which \( f(a) = b \). Consequently, if \( f \) is not a
one-to-one correspondence, we cannot assign to each element \( b \) in the codomain a unique element \( a \) in the domain such that \( f(a) = b \) (because for some \( b \) there is either more than one such \( a \) or no such \( a \)).

A one-to-one correspondence is called invertible since we can define an inverse of this function. A function is not invertible if it is not a one-to-one correspondence, since the inverse of such a function does not exist.

**EXAMPLE 14**

Let \( f \) be the function from \( \{a, b, c\} \) to \( \{1, 2, 3\} \) such that \( f(a) = 2 \), \( f(b) = 3 \), and \( f(c) = 1 \). Is \( f \) invertible, and if it is, what is its inverse?

**Solution:** The function \( f \) is invertible since it is a one-to-one correspondence. The inverse function \( f^{-1} \) reverses the correspondence given by \( f \), so that \( f^{-1}(1) = c \), \( f^{-1}(2) = a \), and \( f^{-1}(3) = b \).

**EXAMPLE 15**

Let \( f \) be the function from the set of integers to the set of integers such that \( f(x) = x + 1 \). Is \( f \) invertible, and if it is, what is its inverse?

**Solution:** The function \( f \) has an inverse since it is a one-to-one correspondence, as we have shown. To reverse the correspondence, suppose that \( y \) is the image of \( x \), so that \( y = x + 1 \). Then \( x = y - 1 \). This means that \( y - 1 \) is the unique element of \( \mathbb{Z} \) that is sent to \( y \) by \( f \). Consequently, \( f^{-1}(y) = y - 1 \).

**EXAMPLE 16**

Let \( f \) be the function from \( \mathbb{Z} \) to \( \mathbb{Z} \) with \( f(x) = x^2 \). Is \( f \) invertible?

**Solution:** Since \( f(-1) = f(1) = 1 \), \( f \) is not one-to-one. If an inverse function were defined, it would have to assign two elements to 1. Hence, \( f \) is not invertible.

**DEFINITION 10.** Let \( g \) be a function from the set \( A \) to the set \( B \) and let \( f \) be a function from the set \( B \) to the set \( C \). The composition of the functions \( f \) and \( g \), denoted by \( f \circ g \), is defined by

\[
(f \circ g)(a) = f(g(a)).
\]

In other words, \( f \circ g \) is the function that assigns to the element \( a \) of \( A \) the element assigned by \( f \) to \( g(a) \). Note that the composition \( f \circ g \) cannot be defined unless the range of \( g \) is a subset of the domain of \( f \). In Figure 7 the composition of functions is shown.
EXAMPLE 17

Let \( g \) be the function from the set \( \{a, b, c\} \) to itself such that \( g(a) = b, \ g(b) = c, \) and \( g(c) = a \). Let \( f \) be the function from the set \( \{a, b, c\} \) to the set \( \{1, 2, 3\} \) such that \( f(a) = 3, \ f(b) = 2, \) and \( f(c) = 1 \). What is the composition of \( f \) and \( g \), and what is the composition of \( g \) and \( f \)?

**Solution:** The composition \( f \circ g \) is defined by \( (f \circ g)(a) = f(g(a)) = f(b) = 2 \),\( (f \circ g)(b) = f(g(b)) = f(c) = 1 \), and \( (f \circ g)(c) = f(g(c)) = f(a) = 3 \).

Note that \( g \circ f \) is not defined, because the range of \( f \) is not a subset of the domain of \( g \).  

EXAMPLE 18

Let \( f \) and \( g \) be the functions from the set of integers to the set of integers defined by \( f(x) = 2x + 3 \) and \( g(x) = 3x + 2 \). What is the composition of \( f \) and \( g \)? What is the composition of \( g \) and \( f \)?

**Solution:** Both the compositions \( f \circ g \) and \( g \circ f \) are defined. Moreover,

\[ (f \circ g)(x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7 \]

and

\[ (g \circ f)(x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11 \]

**Remark:** Note that even though \( f \circ g \) and \( g \circ f \) are defined for the functions \( f \) and \( g \) in Example 18, \( f \circ g \) and \( g \circ f \) are not equal. In other words, the commutative law does not hold for the composition of functions.

When the composition of a function and its inverse is formed, in either order, an identity function is obtained. To see this, suppose that \( f \) is a one-to-one correspondence from the set \( A \) to the set \( B \). Then the inverse function \( f^{-1} \) exists and is a one-to-one correspondence from \( B \) to \( A \). The inverse function reverses the correspondence of the original function, so that \( f^{-1}(b) = a \) when \( f(a) = b \), and \( f(a) = b \) when \( f^{-1}(b) = a \). Hence,

\[ (f^{-1} \circ f)(a) = f^{-1}(f(a)) = f^{-1}(b) = a, \]

and

\[ (f \circ f^{-1})(b) = f(f^{-1}(b)) = f(a) = b. \]
Consequently \( f^{-1} \circ f = \iota_A \) and \( f \circ f^{-1} = \iota_B \), where \( \iota_A \) and \( \iota_B \) are the identity functions on the sets \( A \) and \( B \), respectively. That is, \( (f^{-1})^{-1} = f \).

### THE GRAPHS OF FUNCTIONS

We can associate a set of pairs in \( A \times B \) to each function from \( A \) to \( B \). This set of pairs is called the **graph** of the function and is often displayed pictorially to aid in understanding the behavior of the function.

**DEFINITION 11.** Let \( f \) be a function from the set \( A \) to the set \( B \). The **graph** of the function \( f \) is the set of ordered pairs \( \{(a, b) \mid a \in A \text{ and } f(a) = b\} \).

From the definition, the graph of a function \( f \) from \( A \) to \( B \) is the subset of \( A \times B \) containing the ordered pairs with the second entry equal to the element of \( B \) assigned by \( f \) to the first entry.

**EXAMPLE 19**

Display the graph of the function \( f(n) = 2n + 1 \) from the set of integers to the set of integers.

**Solution:** The graph of \( f \) is the set of ordered pairs of the form \((n, 2n + 1)\) where \( n \) is an integer. This graph is displayed in Figure 8.

**EXAMPLE 20**

Display the graph of the function \( f(x) = x^2 \) from the set of integers to the set of integers.

**Solution:** The graph of \( f \) is the set of ordered pairs of the form \((x, f(x)) = (x, x^2)\) where \( x \) is an integer. This graph is displayed in Figure 9.

---

![Figure 8](image-url)

**FIGURE 8** The Graph of the Function \( f(n) = 2n + 1 \) from \( \mathbb{Z} \) to \( \mathbb{Z} \).

![Figure 9](image-url)

**FIGURE 9** The Graph of \( f(x) = x^2 \) from \( \mathbb{Z} \) to \( \mathbb{Z} \).
Some important functions

Next, we introduce two important functions in discrete mathematics, namely, the floor and ceiling functions. Let \( x \) be a real number. The floor function rounds \( x \) down to the closest integer less than or equal to \( x \), and the ceiling function rounds \( x \) up to the closest integer greater than or equal to \( x \). These functions are often used when objects are counted. They play an important role in the analysis of the number of steps used by procedures to solve problems of a particular size.

Definition 12. The floor function assigns to the real number \( x \) the largest integer that is less than or equal to \( x \). The value of the floor function at \( x \) is denoted by \( \lfloor x \rfloor \). The ceiling function assigns to the real number \( x \) the smallest integer that is greater than or equal to \( x \). The value of the ceiling function at \( x \) is denoted by \( \lceil x \rceil \).

Remark: The floor function is often also called the greatest integer function. It is often denoted by \( \lfloor x \rfloor \).

Example 21

The following are some values of the floor and ceiling functions:

\[
\lfloor \frac{1}{2} \rfloor = 0, \quad \lfloor \frac{3}{2} \rfloor = 1, \quad \lfloor -\frac{1}{2} \rfloor = -1, \quad \lfloor -\frac{3}{2} \rfloor = 0,
\]

\[
\lceil 3.1 \rceil = 3, \quad \lceil 3.1 \rceil = 4, \quad \lceil 7 \rceil = 7, \quad \lceil 7 \rceil = 7.
\]

We display the graphs of the floor and ceiling functions in Figure 10.

The floor and ceiling functions are useful in a wide variety of applications, including those involving data storage and data transmission. Consider the following examples, typical of basic calculations done when database and data communications problems are studied.

![Graphs of the (a) Floor and (b) Ceiling Functions.](image-url)
EXAMPLE 22  
Data stored on a computer disk or transmitted over a data network are usually represented as a string of bytes. Each byte is made up of 8 bits. How many bytes are required to encode 100 bits of data?

Solution:  To determine the number of bytes needed, we determine the smallest integer that is at least as large as the quotient when 100 is divided by 8, the number of bits in a byte. Consequently, \( \lceil 100/8 \rceil = \lceil 12.5 \rceil = 13 \) bytes is required.

EXAMPLE 23  
In asynchronous transfer mode (ATM) (a communications protocol used on backbone networks), data are organized into cells of 53 bytes. How many ATM cells can be transmitted in 1 minute over a connection that transmits data at the rate of 500 kilobits per second?

Solution:  In 1 minute, this connection can transmit \( 500,000 \cdot 60 = 30,000,000 \) bits. Each ATM cell is 53 bytes long, which means that it is \( 53 \cdot 8 = 424 \) bits long. To determine the number of cells that can be transmitted in 1 minute, we determine the largest integer not exceeding the quotient when \( 30,000,000 \) is divided by 424. Consequently, \( \lfloor 30,000,000/424 \rfloor = 70,754 \) ATM cells can be transmitted in 1 minute over a 500 kilobit per second connection.

Table 1, with \( x \) denoting a real number, displays some simple but important properties of the floor and ceiling functions. Since these functions appear so frequently in discrete mathematics, it is useful to look over these identities. Each property in this table can be established using the definitions of the floor and ceiling functions. Properties (1a), (1b), (1c), and (1d) follow directly from these definitions. For example, (1a) states that \( \lfloor x \rfloor = n \) if and only if the integer \( n \) is less than or equal to \( x \) and \( n + 1 \) is larger than \( x \). This is precisely what it means for \( n \) to be the greatest integer not exceeding \( x \), which is the definition of \( \lfloor x \rfloor = n \). Properties (1b), (1c), and (1d) can be established similarly.

We will show that (4a) is true. To show that (4a) is true, suppose that \( \lfloor x \rfloor = n \) where \( n \) is an integer. By (1a) it follows that \( n \leq x < n + 1 \). Adding \( m \) to this inequality shows that \( n + m \leq x + m < n + m + 1 \). Using (1a) again, we see that \( \lfloor x + m \rfloor = n + m = \lfloor x \rfloor + m \).
which is what we wanted to show. We defer establishing the other properties to the exercises.

There are certain types of functions that will be used throughout the text. These include polynomial, logarithmic, and exponential functions. A brief review of the properties of these functions needed in this text is given in Appendix 1. In this book the notation $\log x$ will be used to denote the logarithm to the base 2 of $x$, since 2 is the base that we will usually use for logarithms. We will denote logarithms to the base $b$, where $b$ is any real number greater than 1, by $\log_b x$.

## Exercises

1. Why is $f$ not a function from $\mathbb{R}$ to $\mathbb{R}$ in the following equations?
   a) $f(x) = 1/x$
   b) $f(x) = \sqrt{x}$
   c) $f(x) = \pm \sqrt{x^2 + 1}$

2. Determine whether $f$ is a function from $\mathbb{Z}$ to $\mathbb{R}$ if
   a) $f(n) = \pm n$
   b) $f(n) = \sqrt{n^2 + 1}$
   c) $f(n) = 1/(n^2 - 4)$.

3. Determine whether $f$ is a function from the set of all bit strings to the set of integers if
   a) $f(S)$ is the position of a 0 bit in $S$.
   b) $f(S)$ is the number of 1 bits in $S$.
   c) $f(S)$ is the smallest integer $i$ such that the $i$th bit of $S$ is 1 and $f(S) = 0$ when $S$ is the empty string, the string with no bits.

4. Find the domain and range of the following functions.
   a) the function that assigns to each nonnegative integer its last digit
   b) the function that assigns the next largest integer to a positive integer
   c) the function that assigns to a bit string the number of one bits in the string
   d) the function that assigns to a bit string the number of bits in the string

5. Find the domain and range of the following functions.
   a) the function that assigns to each bit string the difference between the number of ones and the number of zeros
   b) the function that assigns to each bit string twice the number of zeros in that string
   c) the function that assigns the number of bits left over when a bit string is split into bytes (which are blocks of 8 bits)
   d) the function that assigns to each positive integer the largest perfect square not exceeding this integer

6. Find the following values.
   a) $[\frac{1}{2}]$
   b) $[\frac{3}{2}]$
   c) $[-\frac{3}{2}]$
   d) $[-\frac{5}{2}]$
   e) $[3]$
   f) $[-1]$  
   g) $[\frac{1}{2}] + [\frac{3}{2}] + \frac{1}{2}$
   h) $[\frac{1}{2}] + [\frac{3}{2}] + \frac{1}{2}$

7. Find the following values.
   a) $\sqrt{\frac{1}{2}}$
   b) $\sqrt{\frac{3}{2}}$
   c) $-\sqrt{\frac{3}{2}}$
   d) $-\sqrt{\frac{5}{2}}$
   e) $\sqrt{3}$
   f) $-\sqrt{2}$
   g) $\frac{1}{2} + \sqrt{\frac{3}{2}}$
   h) $\frac{1}{2} + \sqrt{\frac{3}{2}}$

8. Determine whether each of the following functions from $(a, b, c, d)$ to itself is one-to-one.
   a) $f(a) = b, f(b) = a, f(c) = c, f(d) = d$
   b) $f(a) = f(b) = f(c) = f(d) = c$
   c) $f(a) = d, f(b) = b, f(c) = c, f(d) = d$

9. Which functions in Exercise 8 are onto?

10. Determine whether each of the following functions from $Z$ to $Z$ is one-to-one.
    a) $f(n) = n - 1$
    b) $f(n) = n^2 + 1$
    c) $f(n) = n^3$
    d) $f(n) = \lfloor n/2 \rfloor$

11. Which functions in Exercise 10 are onto?

12. Give an example of a function from $\mathbb{N}$ to $\mathbb{N}$ that is
    a) one-to-one but not onto,
    b) onto but not one-to-one,
    c) both onto and one-to-one (but different from the identity function),
    d) neither one-to-one nor onto.

13. Give an explicit formula for a function from the set of integers to the set of positive integers that is
    a) one-to-one, but not onto,
    b) onto, but not one-to-one,
    c) one-to-one and onto,
    d) neither one-to-one nor onto.

14. Determine whether each of the following functions is a bijection from $\mathbb{R}$ to $\mathbb{R}$.
    a) $f(x) = -3x + 4$
    b) $f(x) = -3x^2 + 7$
    c) $f(x) = (x + 1)(x + 2)$
    d) $f(x) = x^2 + 1$

15. Determine whether each of the following functions is a bijection from $\mathbb{R}$ to $\mathbb{R}$.
    a) $f(x) = 2x + 1$
    b) $f(x) = x^2 + 1$
    c) $f(x) = x^3$
    d) $f(x) = (x^2 + 1)/(x^2 + 2)$
16. Let $S = \{-1, 0, 2, 4, 7\}$. Find $f(S)$ if
   a) $f(x) = 1$.  b) $f(x) = 2x + 1$.  
   c) $f(x) = \lceil x \rceil$.  d) $f(x) = \lfloor (x^2 + 1)/3 \rfloor$.
17. Let $f(x) = \lfloor x^2/3 \rfloor$. Find $f(S)$ if
   a) $S = \{-2, -1, 0, 1, 2, 3\}$.  b) $S = \{0, 1, 2, 3, 4, 5\}$.  
   c) $S = \{1, 5, 7, 11\}$.  d) $S = \{2, 6, 10, 14\}$.
18. Let $f(x) = 2x$. What is
   a) $f(2)$?  b) $f(N)$?  c) $f(R)$?
19. Suppose that $g$ is a function from $A$ to $B$ and $f$ is a function from $B$ to $C$.
   a) Show that if both $f$ and $g$ are one-to-one functions, then $f \circ g$ is also one-to-one.
   b) Show that if both $f$ and $g$ are onto functions, then $f \circ g$ is also onto.
*20. If $f$ and $g \circ f$ are one-to-one, does it follow that $g$ is one-to-one? Justify your answer.
*21. If $f$ and $g \circ f$ are onto, does it follow that $g$ is onto? Justify your answer.
22. Find $f \circ g$ and $g \circ f$ where $f(x) = x^2 + 1$ and $g(x) = x + 2$ are functions from $R$ to $R$.
23. Find $f + g$ and $fg$ for the functions $f$ and $g$ given in Exercise 22.
24. Let $f(x) = ax + b$ and $g(x) = cx + d$ where $a, b, c, d$ are constants. Determine for which constants $a, b, c,$ and $d$ it is true that $f \circ g = g \circ f$.
25. Show that the function $f(x) = ax + b$ from $R$ to $R$ is invertible, where $a$ and $b$ are constants, with $a \neq 0$, and find the inverse of $f$.
26. Let $f$ be a function from the set $A$ to the set $B$. Let $S$ and $T$ be subsets of $A$. Show that
   a) $f(S \cup T) = f(S) \cup f(T)$.  b) $f(S \cap T) \subseteq f(S) \cap f(T)$.
27. Give an example to show that the inclusion in part (b) in Exercise 26 may be proper.
Let $f$ be a function from the set $A$ to the set $B$. Let $S$ be a subset of $B$. We define the inverse image of $S$ to be the subset of $A$ containing all pre-images of all elements of $S$. We denote the inverse image of $S$ by $f^{-1}(S)$, so that $f^{-1}(S) = \{a \in A \mid f(a) \in S\}$.
28. Let $f$ be the function from $R$ to $R$ defined by $f(x) = x^2$. Find
   a) $f^{-1}(\{1\})$.  b) $f^{-1}(\{0 \leq x < 1\})$.
   c) $f^{-1}(\{x \leq 4\})$.
29. Let $g(x) = |x|$. Find
   a) $g^{-1}(\{0\})$.  b) $g^{-1}(\{-1, 0, 1\})$.
   c) $g^{-1}(\{0 \leq x < 1\})$.
30. Let $f$ be a function from $A$ to $B$. Let $S$ and $T$ be subsets of $B$. Show that
   a) $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$.
   b) $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$.
31. Let $f$ be a function from $A$ to $B$. Let $S$ be a subset of $B$. Show that $f^{-1}(S) = \bar{S}$.
32. Show that $\lfloor x + \frac{1}{2} \rfloor$ is the closest integer to the number $x$, except when $x$ is midway between two integers, when it is the larger of these two integers.
33. Show that $\lfloor x - \frac{1}{2} \rfloor$ is the closest integer to the integer $x$, except when $x$ is midway between two integers, when it is the smaller of these two integers.
34. Show that if $x$ is a real number, then $\lfloor x \rfloor - \lfloor x \rfloor = 1$ if $x$ is not an integer and $\lfloor x \rfloor - \lfloor x \rfloor = 0$ if $x$ is an integer.
35. Show that if $x$ is a real number, then $x - 1 < \lfloor x \rfloor \leq x < x + 1$.
36. Show that if $x$ is a real number and $m$ is an integer, then $\lfloor x + m \rfloor = \lfloor x \rfloor + m$.
37. Show that if $x$ is a real number and $n$ is an integer, then
   a) $x < n$ if and only if $\lfloor x \rfloor < n$.
   b) $n < x$ if and only if $\lfloor x \rfloor < n$.
38. Show that if $x$ is a real number and $n$ is an integer, then
   a) $x \equiv n$ if and only if $\lfloor x \rfloor \equiv n$.
   b) $n \leq x$ if and only if $\lfloor x \rfloor \leq n$.
39. Let $x$ be a real number. Show that $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$.
40. Prove that if $x$ is a real number, then $\lfloor -x \rfloor = -\lfloor x \rfloor$ and $\lfloor -x \rfloor = -\lfloor x \rfloor$.
41. The function INT is found on some calculators, where INT(x) = [x] when x is a nonnegative real number and INT(x) = [x] when x is a negative real number. Show that this INT function satisfies the identity INT(-x) = -INT(x).
42. Let $a$ and $b$ be real numbers with $a < b$. Use the floor and ceiling functions to express the number of integers $n$ that satisfy the inequality $a \leq n \leq b$.
43. Let $a$ and $b$ be real numbers with $a < b$. Use the floor and ceiling functions to express the number of integers $n$ that satisfy the inequality $a < n < b$.
44. How many bytes are required to encode $n$ bits of data where $n$ equals
   a) 4?  b) 10?  c) 500?  d) 3000?
45. How many bytes are required to encode $n$ bits of data where $n$ equals
   a) 7?  b) 17?  c) 1001?  d) 28800?
46. How many ATM cells (described in Example 23) can be transmitted in 10 seconds over a link operating at the following rates?
   a) 128 kilobits per second (1 kilobit = 1000 bits)  
   b) 300 kilobits per second
   c) 1 megabit per second (1 megabit = 1,000,000 bits)
47. Data are transmitted over a particular Ethernet network in blocks of 1500 octets (blocks of 8 bits). How many blocks are required to transmit the following amounts of data over this Ethernet network? (Note that a byte is a synonym for an octet, a kilobyte is 1000 bytes, and a megabyte is 1,000,000 bytes.)
   a) 150 kilobytes of data  
   b) 384 kilobytes of data  
   c) 1.544 megabytes of data  
   d) 43.5 megabytes of data
48. Draw the graph of the function $f(n) = 1 - n^2$ from $Z$ to $Z$.
49. Draw the graph of the function $f(x) = [2x]$ from $R$ to $R$. 

50. Draw the graph of the function \( f(x) = \lfloor x/2 \rfloor \) from \( \mathbb{R} \) to \( \mathbb{R} \).

51. Draw the graph of the function \( f(x) = \lfloor x \rfloor + \lfloor x/2 \rfloor \) from \( \mathbb{R} \) to \( \mathbb{R} \).

52. Draw the graph of the function \( f(x) = \lfloor x \rfloor + \lfloor x/2 \rfloor \) from \( \mathbb{R} \) to \( \mathbb{R} \).

53. Draw graphs of each of the following functions.
   a) \( f(x) = x + \frac{1}{2} \)
   b) \( f(x) = \lfloor 2x \rfloor + 1 \)
   c) \( f(x) = x/3 \)
   d) \( f(x) = \lfloor 1/x \rfloor \)
   e) \( f(x) = \lfloor x - 2 \rfloor + \lfloor x + 2 \rfloor \)
   f) \( f(x) = \lfloor 2x \rfloor / \lfloor x/2 \rfloor \)
   g) \( f(x) = \lfloor x - \frac{1}{2} \rfloor + \frac{1}{2} \)

54. Draw graphs of each of the following functions.
   a) \( f(x) = \lfloor 3x - 2 \rfloor \)
   b) \( f(x) = \lfloor 0.2x \rfloor \)
   c) \( f(x) = \lfloor -1/x \rfloor \)
   d) \( f(x) = \lfloor x^2 \rfloor \)
   e) \( f(x) = \lfloor x/2 \rfloor / \lfloor x/2 \rfloor \)
   f) \( f(x) = \lfloor x/2 \rfloor + \lfloor x/2 \rfloor \)
   g) \( f(x) = \lfloor 2x/2 \rfloor + \frac{1}{2} \)

55. Find the inverse function of \( f(x) = x^3 + 1 \).

56. Suppose that \( f \) is an invertible function from \( Y \) to \( Z \) and \( g \) is an invertible function from \( X \) to \( Y \). Show that the inverse of the composition \( f \circ g \) is given by 
   \( (f \circ g)^{-1} = g^{-1} \circ f^{-1} \).

57. Let \( S \) be a subset of a universal set \( U \). The characteristic function \( f_S \) of \( S \) is the function from \( U \) to the set \( \{0, 1\} \) such that \( f_S(x) = 1 \) if \( x \) belongs to \( S \) and \( f_S(x) = 0 \) if \( x \) does not belong to \( S \). Let \( A \) and \( B \) be sets. Show that for all \( x \)
   a) \( f_{A \cup B}(x) = f_A(x) \cdot f_B(x) \)
   b) \( f_{A \cap B}(x) = f_A(x) + f_B(x) - f_A(x) \cdot f_B(x) \)
   c) \( f_{A}(x) = 1 - f_{A}(x) \)
   d) \( f_{A \setminus B}(x) = f_A(x) + f_B(x) - 2f_A(x)f_B(x) \)

58. Suppose that \( f \) is a function from \( A \) to \( B \), where \( A \) and \( B \) are finite sets with \( |A| = |B| \). Show that \( f \) is one-to-one if and only if it is onto.

A program designed to evaluate a function may not produce the correct value of the function for all elements in the domain of this function. For example, a program may not produce a correct value because evaluating the function may lead to an infinite loop or an overflow.

To study such situations, we use the concept of a partial function. A partial function \( f \) from a set \( A \) to a set \( B \) is an assignment to each element \( a \) in \( A \) a subset \( C_a \) of \( B \), called the domain of \( f \), of a unique element \( b \) in \( B \). We say that \( f \) is undefined for elements in \( A \) that are not in the domain of \( f \). The sets \( A \) and \( B \) are called the domain and codomain of \( f \), respectively.

59. For each of the following partial functions, determine its domain, codomain, domain of definition, and the set of values for which it is undefined. Also, determine whether \( f \) is a partial function or a total function. When the domain of definition of \( f \) equals \( A \), we say that \( f \) is a total function.

60. a) Show that a partial function from \( A \) to \( B \) can be viewed as a function \( f^* \) from \( A \) to \( B \cup \{u\} \) where \( u \) is not an element of \( B \) and
   \[ f^*(a) = \begin{cases} f(a) & \text{if } a \text{ belongs to the domain of definition of } f \\ u & \text{if } f \text{ is undefined at } a \end{cases} \]
   b) Using the construction in (a), find the function \( f^* \) corresponding to each partial function in Exercise 59.

1.7 Sequences and Summations

INTRODUCTION

Sequences are used to represent ordered lists of elements. Sequences are used in discrete mathematics in many ways. They can be used to represent solutions to certain counting problems, as we will see in Chapter 5. They are also an important data structure in computer science. This section contains a review of the concept of a function, as well as the notation used to represent sequences and sums of terms of sequences.

When the elements of an infinite set can be listed, the set is called countable. We will conclude this section with a discussion of both countable and uncountable sets.

SEQUENCES

A sequence is a discrete structure used to represent an ordered list.