The Foundations: Logic, Sets, and Functions

This chapter reviews the foundations of discrete mathematics. Three important topics are covered: logic, sets, and functions. The rules of logic specify the precise meaning of mathematical statements. For instance, the rules give us the meaning of such statements as, "There exists an integer that is greater than 100 that is a power of 2." and, "For every integer $n$ the sum of the positive integers not exceeding $n$ is $n(n + 1)/2." " Logic is the basis of all mathematical reasoning, and it has practical applications to the design of computing machines, to artificial intelligence, to computer programming, to programming languages, and to other areas of computer science.

Much of discrete mathematics is devoted to the study of discrete structures, which are used to represent discrete objects. All discrete structures are built up from sets, which are collections of objects. Examples of discrete structures built up from sets include combinations, which are unordered collections of objects used extensively in counting; relations, which are sets of ordered pairs that represent relationships between objects; graphs, which are sets of vertices and edges that connect vertices; and finite state machines, which are used to model computing machines.

The concept of a function is extremely important in discrete mathematics. A function assigns to each element of a set precisely one element of a set. Such useful structures as sequences and strings are special types of functions. Functions are used to represent the number of steps a procedure uses to solve a problem. The analysis of algorithms uses terminology and concepts related to the growth of functions. Recursive functions, defined by specifying their values at positive integers in terms of their values at smaller positive integers, are used to solve many counting problems.

1.1 Logic

INTRODUCTION

The rules of logic give precise meaning to mathematical statements. These rules are used to distinguish between valid and invalid mathematical arguments. Since a major goal of this book is to teach the reader how to understand and how to construct correct mathematical arguments, we begin our study of discrete mathematics with an introduction to logic.

In addition to its importance in understanding mathematical reasoning, logic has numerous applications in computer science. These rules are used in the design of computer circuits, the construction of computer programs, the verification of the correctness of programs, and in many other ways. We will discuss each of these applications in the following chapters.
PROPOSITIONS

Our discussion begins with an introduction to the basic building blocks of logic—
propositions. A proposition is a statement that is either true or false, but not both.

EXAMPLE 1

All the following statements are propositions.

1. Washington, D.C., is the capital of the United States of America.
2. Toronto is the capital of Canada.
3. \(1 + 1 = 2\).
4. \(2 + 2 = 3\).

Propositions 1 and 3 are true, whereas 2 and 4 are false.

Some sentences that are not propositions are given in the next example.

EXAMPLE 2

Consider the following sentences.

1. What time is it?
2. Read this carefully.
3. \(x + 1 = 2\).
4. \(x + y = z\).

Sentences 1 and 2 are not propositions because they are not statements. Sentences 3
and 4 are not propositions because they are neither true nor false, since the variables in
these sentences have not been assigned values. Various ways to form propositions from
sentences of this type will be discussed in Section 1.3.

Letters are used to denote propositions, just as letters are used to denote variables.
The conventional letters used for this purpose are \(p, q, r, s, \ldots\). The truth value of a
proposition is true, denoted by \(T\), if it is a true proposition and false, denoted by \(F\), if it
is a false proposition.

We now turn our attention to methods for producing new propositions from those
that we already have. These methods were discussed by the English mathematician
George Boole in 1854 in his book *The Laws of Thought*. Many mathematical statements
are constructed by combining one or more propositions. New propositions, called compound
propositions, are formed from existing propositions using logical operators.

**DEFINITION 1.** Let \(p\) be a proposition. The statement

\[\text{"It is not the case that } p\text{."}\]

is another proposition, called the negation of \(p\). The negation of \(p\) is denoted by \(\neg p\).
The proposition \(\neg p\) is read "not \(p\)."
Find the negation of the proposition
“Today is Friday”
and express this in simple English.

Solution: The negation is
“It is not the case that today is Friday.”
This negation can be more simply expressed by
“Today is not Friday.”

Remark: Strictly speaking, sentences involving variable times such as those in Example 3 are not propositions unless a fixed time is assumed. The same holds for variable places unless a fixed place is assumed and for pronouns unless a particular person is assumed.

A truth table displays the relationships between the truth values of propositions.
Truth tables are especially valuable in the determination of the truth values of propositions constructed from simpler propositions. Table 1 displays all possible truth values of a proposition and the corresponding truth values of its negation.

The negation of a proposition can also be considered the result of the operation of the negation operator on a proposition. The negation operator constructs a new proposition from a single existing proposition. We will now introduce the logical operators that are used to form new propositions from two or more existing propositions. These logical operators are also called connectives.

<table>
<thead>
<tr>
<th>TABLE 1</th>
<th>The Truth Table for the Negation of a Proposition.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p )</td>
<td>( \neg p )</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

George Boole (1815–1864). George Boole, the son of a cobbler, was born in Lincoln, England, in November 1815. Because of his family’s difficult financial situation, Boole had to struggle to educate himself while supporting his family. Nevertheless, he became one of the most important mathematicians of the 1800s. Although he considered a career as a clergyman, he decided instead to go into teaching and soon afterward opened a school of his own. In his preparation for teaching mathematics, Boole—unsatisfied with textbooks of his day—decided to read the works of the great mathematicians. While reading papers of the great French mathematician Lagrange, Boole made discoveries in the calculus of variations, the branch of analysis dealing with finding curves and surfaces optimizing certain parameters.

In 1848 Boole published The Mathematical Analysis of Logic, the first of his contributions to symbolic logic. In 1849 he was appointed professor of mathematics at Queen’s College in Cork, Ireland. In 1854 he published The Laws of Thought, his most famous work. In this book Boole introduced what is now called Boolean algebra in his honor. Boole wrote textbooks on differential equations and on difference equations that were used in Great Britain until the end of the nineteenth century. Boole married in 1855: his wife was the niece of the professor of Greek at Queen’s College. In 1864 Boole died from pneumonia, which he contracted as a result of keeping a lecture engagement even though he was soaking wet from a rainstorm.
**DEFINITION 2.** Let \( p \) and \( q \) be propositions. The proposition "\( p \) and \( q \)," denoted by \( p \land q \), is the proposition that is true when both \( p \) and \( q \) are true and is false otherwise. The proposition \( p \land q \) is called the conjunction of \( p \) and \( q \).

The truth table for \( p \land q \) is shown in Table 2. Note that there are four rows in this truth table, one row for each possible combination of truth values for the propositions \( p \) and \( q \).

**EXAMPLE 4**

Find the conjunction of the propositions \( p \) and \( q \) where \( p \) is the proposition "Today is Friday" and \( q \) is the proposition "It is raining today."

*Solution:* The conjunction of these propositions, \( p \land q \), is the proposition "Today is Friday and it is raining today." This proposition is true on rainy Fridays and false on any day that is not a Friday and on Fridays when it does not rain.

**DEFINITION 3.** Let \( p \) and \( q \) be propositions. The proposition "\( p \) or \( q \)," denoted by \( p \lor q \), is the proposition that is false when \( p \) and \( q \) are both false and true otherwise. The proposition \( p \lor q \) is called the disjunction of \( p \) and \( q \).

The truth table for \( p \lor q \) is shown in Table 3.

The use of the connective \( or \) in a disjunction corresponds to one of the two ways the word \( or \) is used in English, namely, in an inclusive way. A disjunction is true when either of the two propositions in it is true or when both are true. For instance, the inclusive or is being used in the statement

"Students who have taken calculus or computer science can take this class."

Here, we mean that students who have taken both calculus and computer science can take the class, as well as the students who have taken just one of the two subjects. On the other hand, we are using the exclusive or when we say

"Students who have taken calculus or computer science, but not both, can enroll in this class."

Here, we mean that students who have taken both calculus and a computer science course cannot take the class. Only those who have taken exactly one of the two courses can take the class.

<table>
<thead>
<tr>
<th>Table 2: The Truth Table for the Conjunction of Two Propositions.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p )</td>
</tr>
<tr>
<td>T</td>
</tr>
<tr>
<td>T</td>
</tr>
<tr>
<td>F</td>
</tr>
<tr>
<td>F</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 3: The Truth Table for the Disjunction of Two Propositions.</th>
</tr>
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<tbody>
<tr>
<td>( p )</td>
</tr>
<tr>
<td>T</td>
</tr>
<tr>
<td>T</td>
</tr>
<tr>
<td>F</td>
</tr>
<tr>
<td>F</td>
</tr>
</tbody>
</table>
Similarly, when a menu at a restaurant states, "Soup or salad comes with an entrée," the restaurant almost always means that customers can have either soup or salad, but not both. Hence, this is an exclusive, rather than an inclusive, or.

What is the disjunction of the propositions \( p \) and \( q \) where \( p \) and \( q \) are the same propositions as in Example 4?

**Solution:** The disjunction of \( p \) and \( q \), \( p \lor q \), is the proposition

"Today is Friday or it is raining today."

This proposition is true on any day that is either a Friday or a rainy day (including rainy Fridays). It is only false on days that are not Fridays when it also does not rain.

As was previously remarked, the use of the connective or in a disjunction corresponds to one of the two ways the word or is used in English, namely, in an inclusive way. Thus, a disjunction is true when either of the two propositions in it is true or when both are true. Sometimes, we use or in an exclusive sense. When the exclusive or is used to connect the propositions \( p \) and \( q \), the proposition "\( p \) or \( q \) (but not both)" is obtained. This proposition is true when \( p \) is true and \( q \) is false, or vice versa, and it is false when both \( p \) and \( q \) are false and when both are true.

**DEFINITION 4.** Let \( p \) and \( q \) be propositions. The exclusive or of \( p \) and \( q \), denoted by \( p \oplus q \), is the proposition that is true when exactly one of \( p \) and \( q \) is true and false otherwise.

The truth table for the exclusive or of two propositions is displayed in Table 4.

We will discuss several other important ways that propositions may be combined.

**DEFINITION 5.** Let \( p \) and \( q \) be propositions. The implication \( p \rightarrow q \) is the proposition that is false when \( p \) is true and \( q \) is false and true otherwise. In this implication \( p \) is called the hypothesis (or antecedent or premise) and \( q \) is called the conclusion (or consequence).

The truth table for the implication \( p \rightarrow q \) is shown in Table 5.
Because implications arise in many places in mathematical reasoning, a wide variety of terminology is used to express \( p \rightarrow q \). Some of the more common ways of expressing this implication are:

- "if \( p \), then \( q \)"
- "\( p \) implies \( q \)"
- "\( p \) if \( q \)"
- "\( p \) only if \( q \)"
- "\( q \) is sufficient for \( p \)"
- "\( q \) if \( p \)"
- "\( q \) whenever \( p \)"
- "\( q \) is necessary for \( p \)"

Note that \( p \rightarrow q \) is false only in the case that \( p \) is true but \( q \) is false, so that it is true when both \( p \) and \( q \) are true, and when \( p \) is false (no matter what truth value \( q \) has).

A useful way to remember that an implication is true when its hypothesis is false is to think of a contract or an obligation. If the condition specified by such a statement is false, no obligation is in force. For example, the statement "If you make more than $25,000, then you must file a tax return" says nothing about someone making less than $25,000. You violate the obligation only if you make more than $25,000 and do not file a return. Similarly, the statement "If a player hits more than 60 home runs, then a bonus of $10 million is awarded" in the contract of a baseball player is violated only when the player hits more than 60 home runs, but the bonus is not awarded. This says nothing if the player hits fewer than 60 home runs.

The way we have defined implications is more general than the meaning attached to implications in the English language. For instance, the implication

"If it is sunny today, then we will go to the beach."

is an implication used in normal language, since there is a relationship between the hypothesis and the conclusion. Further, this implication is considered valid unless it is indeed sunny today, but we do not go to the beach. On the other hand, the implication

"If today is Friday, then \( 2 + 3 = 5 \)."

is true from the definition of implication, since its conclusion is true. (The truth value of the hypothesis does not matter then.) The implication

"If today is Friday, then \( 2 + 3 = 6 \)."

is true every day except Friday, even though \( 2 + 3 = 6 \) is false.

We would not use these last two implications in natural language, since there is no relationship between the hypothesis and the conclusion in either implication. In mathematical reasoning we consider implications of a more general sort than we use in English. The mathematical concept of an implication is independent of a cause-and-effect relationship between hypothesis and conclusion. Our definition of an implication specifies its truth values; it is not based on English usage.

The if-then construction used in many programming languages is different from that used in logic. Most programming languages contain statements such as if \( p \) then \( S \), where \( p \) is a proposition and \( S \) is a program segment (one or more statements to be executed). When execution of a program encounters such a statement, \( S \) is executed if \( p \) is true, but \( S \) is not executed if \( p \) is false, as illustrated in the following example.

**Example 6**

What is the value of the variable \( x \) after the statement

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if 2 + 2 = 4 then x := x + 1
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if \( x \) = 0 before this statement is encountered? (The symbol := stands for assignment. The statement \( x := x + 1 \) means the assignment of the value of \( x + 1 \) to \( x \).)
Solution: Since $2 + 2 = 4$ is true, the assignment statement $x := x + 1$ is executed. Hence, $x$ has the value $0 + 1 = 1$ after this statement is encountered.

We can build up compound propositions using the negation operator and the different connectives defined so far. Parentheses are used to specify the order in which the various logical operators in a compound proposition are applied. In particular, the logical operators in the innermost parentheses are applied first. For instance, \((p \lor q) \land (\neg r)\) is the conjunction of \(p \lor q\) and \(\neg r\). To cut down on the number of parentheses needed, we specify that the negation operator is applied before all other logical operators. This means that \(\neg p \land q\) is the conjunction of \(\neg p\) and \(q\), namely \((\neg p) \land q\), not the negation of the conjunction of \(p\) and \(q\), namely \(\neg (p \land q)\).

There are some related implications that can be formed from \(p \rightarrow q\). The proposition \(q \rightarrow p\) is called the **converse** of \(p \rightarrow q\). The **contrapositive** of \(p \rightarrow q\) is the proposition \(\neg q \rightarrow \neg p\).

Find the converse and the contrapositive of the implication

"If today is Thursday, then I have a test today."

**Solution:** The converse is

"If I have a test today, then today is Thursday."

And the contrapositive of this implication is

"If I do not have a test today, then today is not Thursday."

We now introduce another way to combine propositions.

**Definition 6.** Let \(p\) and \(q\) be propositions. The **biconditional** \(p \iff q\) is the proposition that is true when \(p\) and \(q\) have the same truth values and is false otherwise.

The truth table for \(p \iff q\) is shown in Table 6. Note that the biconditional \(p \iff q\) is true precisely when both the implications \(p \rightarrow q\) and \(q \rightarrow p\) are true. Because of this, the terminology

"\(p\) if and only if \(q\)"

is used for this biconditional. Other common ways of expressing the proposition \(p \iff q\) are: "\(p\) is necessary and sufficient for \(q\)" and "if \(p\) then \(q\), and conversely."

<table>
<thead>
<tr>
<th>(p)</th>
<th>(q)</th>
<th>(p \iff q)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
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<td>T</td>
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<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>
TRANSLATING ENGLISH SENTENCES

There are many reasons to translate English sentences into expressions involving propositional variables and logical connectives. In particular, English (and every other human language) is often ambiguous. Translating sentences into logical expressions removes the ambiguity. Note that this may involve making a set of reasonable assumptions based on the intended meaning of the sentence. Moreover, once we have translated sentences from English into logical expressions we can analyze these logical expressions to determine their truth values, we can manipulate them, and we can use rules of inference (which are discussed in Chapter 3) to reason about them.

To illustrate the process of translating an English sentence into a logical expression, consider the following examples.

**Example 8**

How can the following English sentence be translated into a logical expression?

"You can access the Internet from campus only if you are a computer science major or you are not a freshman."

*Solution:* There are many ways to translate this sentence into a logical expression. Although it is possible to represent the sentence by a single propositional variable, such as \( p \), this would not be useful when analyzing its meaning or reasoning with it. Instead, we will use propositional variables to represent each sentence part and determine the appropriate logical connectives between them. In particular, we let \( a, c, \) and \( f \) represent "You can access the Internet from campus," "You are a computer science major," and "You are a freshman," respectively. Noting that "only if" is one way an implication can be expressed, this sentence can be represented as

\[
a \rightarrow (c \lor \neg f).
\]

**Example 9**

How can the following English sentence be translated into a logical expression?

"You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years old."

*Solution:* There are many ways to translate this sentence into a logical expression. The simplest but least useful way is simply to represent the sentence by a single propositional variable, say, \( p \). Although this is not wrong, doing this would not assist us when we try to analyze the sentence or reason using it. More appropriately, what we can do is to use propositional variables to represent each of the sentence parts and to decide on the appropriate logical connectives between them. In particular, we let \( q, r, \) and \( s \) represent "You can ride the roller coaster," "You are under 4 feet tall," and "You are older than 16 years old," respectively. Then the sentence can be translated to

\[
(r \land \neg s) \rightarrow \neg q.
\]

Of course, there are other ways to represent the original sentence as a logical expression, but the one we have used should meet our needs.

**BOOLEAN SEARCHES**

Logical connectives are used extensively in searches of large collections of information, such as indexes of Web pages. Because these searches employ techniques from propositional logic, they are called **Boolean searches.**
In Boolean searches, the connective AND is used to match records that contain both of two search terms, the connective OR is used to match one or both of two search terms, and the connective NOT (sometimes written as AND NOT) is used to exclude a particular search term. Careful planning of how logical connectives are used is often required when Boolean searches are used to locate information of potential interest. The following example illustrates how Boolean searches are carried out.

Web Page Searching. Most Web search engines support Boolean searching techniques, which usually can help find Web pages about particular subjects. For instance, using Boolean searching to find Web pages about universities in New Mexico, we can look for pages matching NEW AND MEXICO AND UNIVERSITIES. The results of this search will include those pages that contain the three words NEW, MEXICO, and UNIVERSITIES. This will include all of the pages of interest, together with others such as a page about new universities in Mexico. Next, to find pages that deal with universities in New Mexico or Arizona, we can search for pages matching (NEW AND MEXICO OR ARIZONA) AND UNIVERSITIES. (Note: Here the OR operator takes precedence over the AND operator.) The results of this search will include all pages that contain the word UNIVERSITIES and either both the words NEW and MEXICO or the word ARIZONA. Again, pages besides those of interest will be listed. Finally, to find Web pages that deal with universities in Mexico (and not New Mexico), we might first look for pages matching MEXICO AND UNIVERSITIES, but since the results of this search will include pages about universities in New Mexico, as well as universities in Mexico, it might be better to search for pages matching (MEXICO AND UNIVERSITIES) NOT NEW. The results of this search include pages that contain both the words MEXICO and UNIVERSITIES but do not contain the word NEW.

LOGIC AND BIT OPERATIONS

Computers represent information using bits. A bit has two possible values, namely, 0 (zero) and 1 (one). This meaning of the word bit comes from binary digit, since zeros and ones are the digits used in binary representations of numbers. The well-known statistician John Tukey introduced this terminology in 1946. A bit can be used to represent a truth value, since there are two truth values, namely, true and false. As is customarily done, we will use a 1 bit to represent true and a 0 bit to represent false. That is, 1 represents T (true), 0 represents F (false). A variable is called a Boolean variable if its value is either true or false. Consequently, a Boolean variable can be represented using a bit.

Computer bit operations correspond to the logical connectives. By replacing true by a one and false by a zero in the truth tables for the operators $\land$, $\lor$, and $\oplus$, the tables shown in Table 7 for the corresponding bit operations are obtained. We will also use the notation OR, AND, and XOR for the operators $\lor$, $\land$, and $\oplus$, as is done in various programming languages.

Information is often represented using bit strings, which are sequences of zeros and ones. When this is done, operations on the bit strings can be used to manipulate this information.

**DEFINITION 7.** A bit string is a sequence of zero or more bits. The length of this string is the number of bits in the string.
### TABLE 7

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$x \lor y$</th>
<th>$x \land y$</th>
<th>$x \oplus y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<tr>
<td>0</td>
<td>1</td>
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<td>1</td>
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<td>0</td>
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</table>

**Example 11**

101010011 is a bit string of length nine.

We can extend bit operations to bit strings. We define the **bitwise OR**, **bitwise AND**, and **bitwise XOR** of two strings of the same length to be the strings that have as their bits the OR, AND, and XOR of the corresponding bits in the two strings, respectively. We use the symbols $\lor$, $\land$, and $\oplus$ to represent the bitwise OR, bitwise AND, and bitwise XOR operations, respectively. We illustrate bitwise operations on bit strings with the following example.

**Example 12**

Find the bitwise OR, bitwise AND, and bitwise XOR of the bit strings 01 1011 0110 and 11 0001 1101. (Here, and throughout this book, bit strings will be split into blocks of four bits to make them easier to read.)

**Solution:** The bitwise OR, bitwise AND, and bitwise XOR of these strings are obtained by taking the OR, AND, and XOR of the corresponding bits, respectively. This gives us

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Historical Note: There were several other suggested words for a binary digit, including *binit* and *bigit*, that never were widely accepted. The adoption of the word *bit* may be due to its meaning as a common English word. For an account of Tukey's coinage of the word *bit*, see the April 1984 issue of *Annals of the History of Computing*.

John Wilder Tukey (born 1915). Tukey, born in New Bedford, Massachusetts, was an only child. His parents, both teachers, decided home schooling would best develop his potential. His formal education began at Brown University, where he studied mathematics and chemistry. He received a master's degree in chemistry from Brown and continued his studies at Princeton University, changing his field of study from chemistry to mathematics. He received his Ph.D. from Princeton in 1939 for work in topology, when he was appointed an instructor in mathematics at Princeton. With the start of World War II, he joined the Fire Control Research Office, where he began working in statistics. Tukey found statistical research to his liking and impressed several leading statisticians with his skills. In 1945, at the conclusion of the war, Tukey returned to the mathematics department at Princeton as a professor of statistics, and he also took a position at AT&T Bell Laboratories. Tukey founded the Statistics Department at Princeton in 1966 and was its first chairman. Tukey made significant contributions to many areas of statistics, including the analysis of variance, the estimation of spectra of time series, inferences about the values of a set of parameters from a single experiment, and the philosophy of statistics. However, he is best known for his invention, with J. W. Cooley, of the fast Fourier transform.

Tukey contributed his insight and expertise by serving on the President's Science Advisory Committee. He chaired several important committees dealing with the environment, education, and chemicals and health. He also served on committees working on nuclear disarmament. Tukey has received many awards, including the National Medal of Science.
Exercises

1. Which of the following sentences are propositions? What are the truth values of those that are propositions?
   a) Boston is the capital of Massachusetts.
   b) Miami is the capital of Florida.
   c) $2 + 3 = 5$.
   d) $5 + 7 = 10$.
   e) $x + 2 = 11$.
   f) Answer this question.
   g) $x + y = y + x$ for every pair of real numbers $x$ and $y$.

2. Which of the following are propositions? What are the truth values of those that are propositions?
   a) Do not pass go.
   b) What time is it?
   c) There are no black flies in Maine.
   d) $4 + x = 5$.
   e) $x + 1 = 5$ if $x = 1$.
   f) $x + y = y + z$ if $x = y$.

3. What is the negation of each of the following propositions?
   a) Today is Thursday.
   b) There is no pollution in New Jersey.
   c) $2 + 1 = 3$.
   d) The summer in Maine is hot and sunny.

4. Let $p$ and $q$ be the propositions
   $p$: I bought a lottery ticket this week.
   $q$: I won the million dollar jackpot on Friday.

   Express each of the following propositions as an English sentence.
   a) $\neg p$
   b) $p \lor q$
   c) $p \rightarrow q$
   d) $p \land q$
   e) $p \equiv q$
   f) $\neg p \rightarrow q$
   g) $\neg p \land \neg q$
   h) $\neg p \lor (p \land q)$

5. Let $p$ and $q$ be the propositions
   $p$: It is below freezing.
   $q$: It is snowing.

   Write the following propositions using $p$ and $q$ and logical connectives.
   a) It is below freezing and snowing.
   b) It is below freezing but not snowing.
   c) It is not below freezing and it is not snowing.
   d) It is either snowing or below freezing (or both).
   e) If it is below freezing, it is also snowing.
   f) It is either below freezing or it is snowing, but it is not snowing if it is below freezing.
   g) That it is below freezing is necessary and sufficient for it to be snowing.

6. Let $p$, $q$, and $r$ be the propositions
   $p$: You have the flu.
   $q$: You miss the final examination.
   $r$: You pass the course.

   Express each of the following propositions as an English sentence.
   a) $p \rightarrow q$
   b) $\neg q \leftrightarrow r$
   c) $q \rightarrow r$
   d) $p \lor q \lor r$
   e) $(p \rightarrow r) \lor (q \rightarrow r)$
   f) $(p \land q) \lor (q \land r)$

7. Let $p$ and $q$ be the propositions
   $p$: You drive over 65 miles per hour.
   $q$: You get a speeding ticket.

   Write the following propositions using $p$ and $q$ and logical connectives.
   a) You do not drive over 65 miles per hour.
   b) You drive over 65 miles per hour, but you do not get a speeding ticket.
   c) You will get a speeding ticket if you drive over 65 miles per hour.
   d) If you do not drive over 65 miles per hour, then you will not get a speeding ticket.
   e) Driving over 65 miles per hour is sufficient for getting a speeding ticket.
   f) You get a speeding ticket, but you do not drive over 65 miles per hour.
   g) Whenever you get a speeding ticket, you are driving over 65 miles per hour.

8. Let $p$, $q$, and $r$ be the propositions
   $p$: You get an A on the final exam.
   $q$: You do every exercise in this book.
   $r$: You get an A in this class.

   Write the following propositions using $p$, $q$, and $r$ and logical connectives.
a) You get an A in this class, but you do not do every exercise in this book.
b) You get an A on the final, you do every exercise in this book, and you get an A in this class.
c) To get an A in this class, it is necessary for you to get an A on the final.
d) You get an A on the final, but you don’t do every exercise in this book; nevertheless, you get an A in this class.
e) Getting an A on the final and doing every exercise in this book is sufficient for getting an A in this class.
f) You will get an A in this class if and only if you either do every exercise in this book or you get an A on the final.

9. Determine whether each of the following implications is true or false.
a) If $1 + 1 = 2$, then $2 + 2 = 5$.
b) If $1 + 1 = 3$, then $2 + 2 = 4$.
c) If $1 + 1 = 3$, then $2 + 2 = 5$.
d) If pigs can fly, then $1 + 1 = 3$.
e) If $1 + 1 = 3$, then God exists.
f) If $1 + 1 = 3$, then pigs can fly.
g) If $1 + 1 = 2$, then pigs can fly.
h) If $2 + 2 = 4$, then $1 + 2 = 3$.

10. For each of the following sentences, determine whether an inclusive or or an exclusive or is intended. Explain your answer.
a) Experience with C++ or Java is required.
b) Lunch includes soup or salad.
c) To enter the country you need a passport or a voter registration card.
d) Publish or perish.

11. For each of the following sentences, state what the sentence means if the or is an inclusive or (that is, a disjunction) versus an exclusive or. Which of these meanings of or do you think is intended?
a) To take discrete mathematics, you must have taken calculus or a course in computer science.
b) When you buy a new car from Acme Motor Company, you get $2000 back in cash or a 2% car loan.
c) Dinner for two includes two items from column A or three items from column B.
d) School is closed if more than 2 feet of snow falls or if the wind chill is below -100.

12. An ancient Sicilian legend says that the barber in a remote town who can be reached only by traveling a dangerous mountain road shaves those people, and only those people, who do not shave themselves. Can there be such a barber?

13. Each inhabitant of a remote village always tells the truth or always lies. A villager will only give a “Yes” or a “No” response to a question a tourist asks. Suppose you are a tourist visiting this area and come to a fork in the road. One branch leads to the ruins you want to visit; the other branch leads deep into the jungle. A villager is standing at the fork in the road. What one question can you ask the villager to determine which branch to take?

14. An explorer is captured by a group of cannibals. There are two types of cannibals—those who always tell the truth and those who always lie. The cannibals will barbecue the explorer unless he can determine whether a particular cannibal always lies or always tells the truth. He is allowed to ask the cannibal exactly one question.
a) Explain why the question “Are you a liar?” does not work.
b) Find a question that the explorer can use to determine whether the cannibal always lies or always tells the truth.

15. Write each of the following statements in the form “if $p$, then $q$” in English. (Hint: Refer to the list of common ways to express implications listed in this section.)
a) It snows whenever the wind blows from the northeast.
b) The apple trees will bloom if it stays warm for a week.
c) That the Pistons win the championship implies that they beat the Lakers.
d) It is necessary to walk 8 miles to get to the top of Long’s Peak.
e) To get tenure as a professor, it is sufficient to be world-famous.
f) If you drive more than 400 miles, you will need to buy gasoline.
g) Your guarantee is good only if you bought your CD player less than 90 days ago.

16. Write each of the following statements in the form “if $p$ then $q$” in English. (Hint: Refer to the list of common ways to express implications listed in this section.)
a) I will remember to send you the address only if you send me an e-mail message.
b) To be a citizen of this country, it is sufficient that you were born in the United States.
c) If you keep your textbook, it will be a useful reference in your future courses.
d) The Red Wings will win the Stanley Cup if their goalie plays well.
e) That you get the job implies that you had the best credentials.
f) The beach erodes whenever there is a storm.
g) It is necessary to have a valid password to log on to the server.

17. Write each of the following propositions in the form “$p$ if and only if $q$” in English.
a) If it is hot outside you buy an ice cream cone, and if you buy an ice cream cone it is hot outside.
b) For you to win the contest it is necessary and sufficient that you have the only winning ticket.
c) You get promoted only if you have connections, and you have connections only if you get promoted.
1.1 Exercises

18. Write each of the following propositions in the form "p if and only if q" in English.
   a) For you to get an A in this course, it is necessary and sufficient that you learn how to solve discrete mathematics problems.
   b) If you can type 100 words per minute, you will be able to type速度快
   c) If it rains, the street is wet.
   d) You can see the wizard only if the wizard is not in.
   e) The trains run late on exactly those days when it is cold.

19. State the converse and contrapositive of each of the following implications.
   a) If it rains, the street is wet.
   b) If you can type 100 words per minute, you will be able to type速度快
   c) If it rains, the street is wet.
   d) You can see the wizard only if the wizard is not in.

20. State the converse and contrapositive of each of the following implications.
   a) If it rains, the street is wet.
   b) If you can type 100 words per minute, you will be able to type速度快
   c) If it rains, the street is wet.
   d) You can see the wizard only if the wizard is not in.

21. Construct a truth table for each of the following compound propositions.
   a) p ∧ ¬p
   b) p ∨ ¬p
   c) (p ∨ ¬q) → q
   d) (p ∨ q) → (p ∧ q)
   e) (p → q) ↔ (¬q → r)
   f) (¬p ↔ q) ↔ (q → r)

22. Construct a truth table for each of the following compound propositions.
   a) p ⊕ p
   b) p ⊕ ¬p
   c) p ⊕ q
   d) (p ⊕ q) ∨ (¬p ⊕ ¬q)
   e) (p ⊕ q) ∧ (¬p ⊕ ¬q)
   f) (p ⊕ q) ∧ (p ⊕ q)

23. Construct a truth table for each of the following compound propositions.
   a) p → ¬q
   b) ¬p ↔ q
   c) (p → q) ∨ (¬p → q)
   d) (p → q) ∨ (¬p → q)
   e) (p → q) ∨ (¬p → q)
   f) (¬p ↔ q) ↔ (p → q)

24. Construct a truth table for each of the following compound propositions.
   a) (p ∨ q) ∨ r
   b) (p ∨ q) ∨ r
   c) (p ∨ q) ∨ r
   d) (p ∨ q) ∨ r
   e) (p ∨ q) ∨ r
   f) (p ∨ q) ∨ r

25. Construct a truth table for each of the following compound propositions.
   a) p → (¬q ∨ r)
   b) ¬p → (q → r)
   c) (p → q) ∨ (¬p → r)
   d) (p → q) ∨ (¬p → r)
normally. If the system is operating normally, the kernel is functioning. The kernel is not functioning or the system is in interrupt mode. If the system is not in multiuser state, then it is in interrupt mode. The system is not in interrupt mode.

36. Are the following specifications consistent? "If the file system is not locked, then new messages will be queued. If the file system is not locked, then the system is functioning normally, and conversely. If new messages are not queued, then they will be sent to the message buffer. If the file system is not locked, then new messages will be sent to the message buffer. New messages will not be sent to the message buffer."

37. What Boolean search would you use to look for Web pages about beaches in New Jersey? What if you wanted to find Web pages about beaches on the isle of Jersey (in the English Channel)?

38. What Boolean search would you use to look for Web pages about hiking in West Virginia? What if you wanted to find Web pages about hiking in Virginia, but not in West Virginia?

Exercises 39–42 are puzzles that can be solved by translating statements into logical expressions and reasoning from these expressions using truth tables.

39. Steve would like to determine the relative salaries of Fred, Maggie, and Janice from what Steve knows? If so, who is paid the most and who the least? Explain your reasoning.

40. Five friends have access to a chat room. Is it possible to determine who is chatting if the following information is known? Either Kevin or Heather, or both, are chatting. Either Randy or Vijay, but not both, are chatting. If Abby is chatting, so is Randi. Vijay and Kevin are either both chatting or neither is. If Heather is chatting, then so are Abby and Kevin. Explain your reasoning.

41. A detective has interviewed four witnesses to a crime. From the stories of the witnesses the detective has concluded that if the butler is telling the truth then so is the cook; the cook and the gardener cannot both be telling the truth; the gardener and the handyman are not both lying; and if the handyman is telling the truth then the cook is lying. For each of the four witnesses, can the detective determine whether that person is telling the truth or lying? Explain your reasoning.

42. Four friends have been identified as suspects for an unauthorized access into a computer system. They have made statements to the investigating authorities. Alice said "Carlos did it." John said "I did not do it." Carlos said "Diana did it." Diana said "Carlos lied when he said that I did it."

a) If the authorities also know that exactly one of the four suspects is telling the truth, who did it? Explain your reasoning.

b) If the authorities also know that exactly one is lying, who did it? Explain your reasoning.

1.2 Propositional Equivalences

INTRODUCTION

An important type of step used in a mathematical argument is the replacement of a statement with another statement with the same truth value. Because of this, methods that produce propositions with the same truth value as a given compound proposition are used extensively in the construction of mathematical arguments.

We begin our discussion with a classification of compound propositions according to their possible truth values.

DEFINITION 1. A compound proposition that is always true, no matter what the truth values of the propositions that occur in it, is called a tautology. A compound proposition that is always false is called a contradiction. Finally, a proposition that is neither a tautology nor a contradiction is called a contingency.

Tautologies and contradictions are often important in mathematical reasoning. The following example illustrates these types of propositions.