b) If \( A \) is true, then both sides are logically equivalent to \( \exists x P(x) \). If \( A \) is false, the left-hand side is clearly false. Furthermore, for every \( x \), \( P(x) \land A \) is false, so \( \exists x (P(x) \land A) \) is false. Hence, the two sides are logically equivalent.

41. To show these are not logically equivalent, let \( P(x) \) be the statement “\( x \) is positive,” and let \( Q(x) \) be the statement “\( x \) is negative.” With universe of discourse the set of integers. Then \( \exists x P(x) \land \exists x Q(x) \) is true, but \( \exists x (P(x) \land Q(x)) \) is false.

43. a) Suppose that \( \forall x (P(x) \land \exists y Q(x)) \) is true. Then \( P(x) \) is true for all \( x \) and there is an element \( y \) for which \( Q(y) \) is true. Since \( P(x) \land Q(y) \) is true for all \( x \) and there is a \( y \) for which \( Q(y) \) is true, \( \forall x \exists y (P(x) \land Q(y)) \) is true. Conversely, suppose that the second proposition is true. Let \( x \) be an element in the universe of discourse. There is a \( y \) such that \( Q(y) \) is true, so \( \exists y Q(y) \) is true. Since \( \forall x P(x) \) is also true, it follows that the first proposition must hold.

b) Suppose that \( \forall x (P(x) \lor \exists y Q(x)) \) is true. Then either \( P(x) \) is true for all \( x \), or there exists a \( y \) for which \( Q(y) \) is true. In the former case, \( P(x) \lor Q(y) \) is true for all \( x \), so \( \forall x Q(x) \lor (P(x) \land Q(y)) \) is true. In the latter case, \( Q(y) \) is true for a particular \( y \), so \( P(x) \lor Q(y) \) is true for all \( x \) and consequently \( \forall x Q(x) \lor Q(y) \) is true. Conversely, suppose that the second proposition is true. If \( P(x) \) is true for all \( x \), then the first proposition is true. If not, \( P(x) \) is false for some \( x \), and for this \( x \) there must be a \( y \) such that \( P(x) \lor Q(y) \) is true. Hence, \( Q(y) \) must be true, so \( \exists y Q(y) \) is true. It follows that the first proposition must hold.

45. a) True
b) False, unless the universe of discourse consists of just one element
c) True

47. \( \exists x (P(x) \land \forall y ((P(x) \land P(y)) \rightarrow x = y) \)

49. We will show how an expression can be put into prenex normal form (PNF) if subexpressions in it can be put into PNF. Then, working from the outside in, any expression can be put in PNF. (To formalize the argument, it is necessary to use the method of mathematical induction for sets that will be discussed in Section 3.3.) By Exercise 29 of Section 1.2, we can assume that the proposition uses only \( \lor \) and \( \land \) as logical connectives. Now consider any proposition with no quantifiers already in PNF. (This is the basis case of the argument.) Now suppose that the proposition is of the form \( Q(x) \land P(x) \), where \( Q \) is a quantifier. Since \( P(x) \) is a shorter expression than the original proposition, we can put it into PNF. Then \( Q(x) \) followed by this PNF is again in PNF and is equivalent to the original proposition. Next, suppose that the proposition is of the form \( \neg P \). If \( P \) is already in PNF, we slide the negation sign past all the quantifiers using the equivalences in Table 3. Finally, assume that proposition is of the form \( P \lor Q \), where each of \( P \) and \( Q \) is in PNF. If only one of \( P \) and \( Q \) has quantifiers, then we can use Exercise 38 to bring the quantifier in front of both. If both \( P \) and \( Q \) have quantifiers, we can use Exercise 37, Exercise 42, or Exercise 43b to rewrite \( P \lor Q \) with two quantifiers preceding the disjunction of a proposition of the form \( R \lor S \), and then put \( R \lor S \) into PNF.

51. \( \forall x \exists y (|x - d| < \delta \land |f(x) - L| \geq \epsilon) \)
53. \( \forall x \exists y (n > N \land |a_n - L| \geq \epsilon) \)
55. \( \forall n \exists n (> N \land L - \epsilon) \land \exists n (> N \land L + \epsilon) \)

SECTION 1.4

1. a) \( \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\} \)
   b) \( \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\} \)
   c) \( \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\} \)
   d) \( \emptyset \)
3. a) Yes b) No c) No d) No e) No f) No
5. a) Yes b) No c) Yes d) No e) Yes f) No
7. a) True b) True c) False d) True e) True f) False
9. Suppose that \( x \in A \). Since \( A \subseteq B \), this implies that \( x \in B \). Since \( B \subseteq C \), we see that \( x \in C \). Since \( x \in A \) implies that \( x \in C \), it follows that \( A \subseteq C \).
11. a) 1 b) 1 c) 2 d) 3
13. a) \( \{1, 2\} \)
   b) \( \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\} \)
   c) \( \{1, 2\} \)
15. a) 8 b) 16 c) 2
17. a) \( \{(a, y), (b, y), (c, y), (d, y), (a, z), (b, z), (c, z), (d, z)\} \)
   b) \( \{(y, a), (y, b), (y, c), (y, d), (z, a), (z, b), (z, c), (z, d)\} \)
19. The set of triples \( (a, b, c) \), where \( a \) is an airline and \( b \) and \( c \) are cities.
21. \( \emptyset \times A = \{(x, y) \mid x \in \emptyset \text{ and } y \in A\} = \emptyset = \{(x, y) \mid x \in A \text{ and } y \in \emptyset\} = A \times \emptyset \)
23. \( mn \)
25. We must show that \( \{(a, b), \{c, d\}\} = \{(a, c), (b, d)\} \) if and only if \( a = c \) and \( b = d \). The “if” part is immediate. So assume these two sets are equal. First, consider the case when \( a \neq b \). Then \( \{(a, b), \{c, d\}\} \) contains exactly two elements, one of which contains one element. Thus, \( \{(a, b), \{c, d\}\} \) must have the same property, so \( c = d \) and \( \{c\} \) contains exactly one element. Hence, \( \{a, c\} \) must have the same property, so \( c = d \) and \( \{c\} \) contains exactly one element. Thus, \( \{a, c\} \) implies that \( a = c \). Also, the two-element sets \( \{a, b\} \) and \( \{c, d\} \) must be equal. Since \( a = c \) and \( a \neq b \), it follows that \( b = d \).
27. Let \( S = \{a_1, a_2, \ldots, a_n\} \). Represent each subset of \( S \) with a bit string of length \( n \), where the \( i \)th bit is 1 if and only if \( a_i \in S \). To generate all subsets of \( S \), list all \( 2^n \) bit strings of length \( n \) (for instance, in increasing order), and write down the corresponding subsets.


SECTION 1.5

1. a) The set of students who live within 1 mile of school and who walk to classes.
b) The set of students who live within 1 mile of school or who walk to classes (or who do both).
c) The set of students who live within 1 mile of school but do not walk to classes.
d) The set of students who walk to classes but live more than 1 mile away from school.

3. a) \{0, 1, 2, 3, 4, 5, 6\} b) \{3\} c) \{1, 2, 4, 5\} d) \{0, 6\}

5. \(\overline{A} = \{x \mid \neg(x \in A)\} = \{x \mid \neg(\neg x \in A)\} = \{x \mid x \in \overline{A}\} = A\).

7. a) \(A \cup B = \{x \mid x \in A \lor x \in B\}\) 
   \(= \{x \mid x \in B \lor x \in A\} = B \cup A\); 
b) \(A \cap B = \{x \mid x \in A \land x \in B\}\) 
   \(= \{x \mid x \in B \land x \in A\} = B \cap A\).

9. a) \(x \in (A \cup B) \iff x \notin (A \cap B)\) 
   \(\iff x \in A \lor x \in B \iff x \notin A \land x \notin B\) 
   \(\iff x \in A \land x \notin B \iff x \in A \land x \notin B\) 
   \(\iff x \in A \land x \notin B \iff x \in A \land x \notin B\) 
   \(\iff x \in A \land x \notin B \iff x \in A \land x \notin B\).

b) 

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>A \cup B</th>
<th>A \cap B</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

11. a) \(x \in \overline{A} \land \overline{B} \land \overline{C} \iff x \notin A \land x \notin B \land x \notin C\) 
   \(x \notin A \land x \notin B \land x \notin C \iff x \in \overline{A} \land x \in \overline{B} \land x \in \overline{C}\).

b) 

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>A \land B \land C</th>
<th>A \land B \land \overline{C}</th>
<th>A \land \overline{B} \land C</th>
<th>\overline{A} \land B \land C</th>
<th>\overline{A} \land \overline{B} \land C</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

13. Both sides equal \(\{x \mid x \in A \land x \notin B\}\).

15. a) \(x \in A \cup (B \cup C) \iff (x \in A) \lor (x \in B \lor x \in C) \iff (x \in A) \lor (x \in B) \lor (x \in C) \iff x \in (A \cup B) \cup C\)
0110 ∧ 01 1110 1010 0000 1100 0111 0111 = 01 1110 0010 0000 1000 0110 0110, representing
\{b, c, d, e, i, o, t, u, v, w, x, y, z\}
\(d)\ 11 1100 0000 0000 0000 0000 0000 \lor 01 1110 1000 0000 0101 0101 1000 0000 0111 \lor 06 0110 0110 0001 1010 0010 1000 0010 0111 = 11 1110 1110 0001 1100 0111 0111, representing
\{a, b, c, d, e, g, h, i, n, o, p, t, u, v, w, x, y, z\}

45. a) \{1, 2, 3, \{1, 2, 3\}\}
   b) \{2\}
   c) \{2\}, \{2\}
   d) \{2\}, \{2\}, \{\{2\}\}

47. a) \{3, a, 3, b, 1, c, 4, d\}
   b) \{2, a, 2, b\}
   c) \{1, a, 1, c\}
   d) \{1, b, 4, d\}
   e) \{5, a, 5, b, 1, c, 4, d\}

49. \(\bar{F} = \{0.4, Alice, 0.6 Fred, 0.9 Oscar, 0.5 Rita\}, \bar{R} = \{0.6 Alice, 0.2 Brian, 0.8 Fred, 0.1 Oscar, 0.3 Rita\}\)

51. \(F \cap R = \{0.4, Alice, 0.8 Brian, 0.2 Fred, 0.1 Oscar, 0.5 Rita\}\)

SECTION 1.6

1. a) \(f(0)\) is not defined.
   b) \(f(x)\) is not defined for \(x < 0\).
   c) \(f(x)\) is not well-defined since there are two distinct values assigned to each \(x\).

3. a) not a function
   b) a function
   c) not a function

5. a) the set of integers
    b) the set of even nonnegative integers
    c) the set of nonnegative integers not exceeding 7
    d) the set of squares of integers = \{0, 1, 4, 9, 16, \ldots\}\)

7. a) 1  b) 0  c) 0  d) -1  e) 3  f) -1  g) 2  h) 1

9. only the function in part (a)

11. only the functions in parts (a) and (d)

13. a) the function \(f(x)\) with \(f(x) = 3x + 1\) when \(x \geq 0\)
    and \(f(x) = -3x + 2\) when \(x < 0\)
    b) \(f(x) = |x| + 1\)
    c) the function \(f(x)\) with \(f(x) = 2x + 1\) when \(x \geq 0\)
    and \(f(x) = -2x\) when \(x < 0\)
    d) \(f(x) = x^2 + 1\)

15. a) Yes  b) No
c) Yes  d) No
d) Yes  e) No

17. a) \(f(S) = \{0, 1, 3\}\)
    b) \(f(S) = \{0, 1, 3, 5, 8\}\)
    c) \(f(S) = \{0, 16, 40\}\)
    d) \(f(S) = \{1, 12, 33, 65\}\)

19. a) Let \(x\) and \(y\) be distinct elements of \(A\). Since \(g\) is one-to-one, \(g(x)\) and \(g(y)\) are distinct elements of \(B\). Since \(f\) is one-to-one, \(f(g(x)) = (f \circ g)(x)\) and \(f(g(y)) = (f \circ g)(y)\) are distinct elements of \(C\). Hence, \(f \circ g\) is one-to-one.

b) Let \(y \in C\). Since \(f\) is onto, \(y = f(b)\) for some \(b \in B\). Now since \(g\) is onto, \(b = g(c)\) for some \(x \in A\). Hence, \(y = (f \circ g)(x) = (f \circ g)(x)\).

It follows that \(f \circ g\) is onto.

21. Denote \(A = \{\{a\}\}, B = \{\{b, c\}\}, and C = \{\{d\}\}\). Let \(g(a) = b\), \(f(b) = d\), and \(f(c) = d\).

23. \((f + g)(x) = x^2 + x - 3\), \((f \circ g)(x) = x^3 + 2x^2 + x + 2\).

25. \(f\) is one-to-one since \(f(x_1) = f(x_2) \iff x_1 = x_2\)

27. \(f(\{1\}) = a\), \(f(\{2\}) = a\). Let \(S = \{1\}\) and \(T = \{2\}\).

\(f(S \cap T) = f(\{1\}) = a\), \(f(S \cap T) = f(\{2\}) = a\), \(f(S\cap T) = \{a\} \cap \{a\} = \{a\}\).

29. a) \(\{x | 0 \leq x < 1\}\)
    b) \(\{x | -1 \leq x < 2\}\)
    c) \(\emptyset\)

31. \(f^{-1}(S) = \{x \in A | f(x) \notin S\} = \{x \in A | f(x) \in S\} = f^{-1}(S)\)

33. \(x = |x| + \varepsilon\), where \(\varepsilon\) is a real number with \(0 \leq \varepsilon < 1\).

35. Write the real number \(x\) as \(|x| + \varepsilon\), where \(\varepsilon\) is a real number with \(0 \leq \varepsilon < 1\). Since \(x = |x| + \varepsilon\), it follows that \(0 \leq |x| < 1\). The first two inequalities, \(|x| < 1\) and \(|x| = |x|\), are valid for the other two inequalities, write \(x = |x| + \varepsilon\), where \(0 \leq \varepsilon < 1\). Then \(0 \leq |x| < 1\) and the desired inequality follows.

37. a) If \(x < n\), since \(|x| < n\), it follows that \(|x| < n\). Suppose that \(x \geq n\). By the definition of the floor function, it follows that \(|x| \geq n\). This means that if \(|x| < n\), then \(x < n\).

b) If \(n < x\), then since \(x = |x| + \varepsilon\), it follows that \(n \leq |x|\).

39. Suppose that \(N \leq x < N + 1\). If \(x = n + \varepsilon\), \(|x| = N\) and \(|x + \frac{1}{2}| = N + 1\), so \(|x| = |x + \frac{1}{2}| = N + 1\). If \(x < N + \frac{1}{2}\), then \(|x| = 2N\) and \(|x + \frac{1}{2}| = N\), and again the identity follows.

41. Assume that \(x = 0\). The left-hand side is \(-|x|\) and the right-hand side is \(-|x|\). If \(x\) is an integer, then both sides equal \(-x\). Otherwise, let \(x = n + \varepsilon\), where \(n\) is a natural number and \(\varepsilon\) is a real number with \(0 \leq \varepsilon < 1\). Then \(-|x| = \varepsilon - n\) and \(-|x| = \varepsilon + n\) also. When \(x < 0\), the equation also holds since it can be obtained by substituting \(-x\) for \(x\).

43. \(|B| - |A| - 1\)

45. a) 1  b) 3  c) 126  d) 3600

47. a) 100  b) 256  c) 1030  d) 30,200
S-10  Solutions to Odd-Numbered Exercises

49.  

51.  

53. a)  

b)  

c)  

d)  

e)  

55.  

57.  

59.  

SECTION 1  
1. a)  

3. a)  

10.
Solutions to Odd-Numbered Exercises

55. \( f^{-1}(y) = (y - 1)^{1/3} \)

57. a) \( f_{A \cap B}(x) = 1 \iff x \in A \cap B \iff x \in A \) and \( x \in B \iff f_A(x) = 1 \) and \( f_B(x) = 1 \iff f_A(x)f_B(x) = 1 \)
   b) \( f_{A \cup B}(x) = 1 \iff x \in A \cup B \iff x \in A \) or \( x \in B \iff f_A(x) = 1 \) or \( f_B(x) = 1 \iff f_A(x) + f_B(x) - f_A(x)f_B(x) = 1 \)
   c) \( f_{A \setminus B}(x) = 1 \iff x \in A \setminus B \iff x \in A \) \( \iff f_A(x) = 0 \iff 1 - f_A(x) = 1 \)
   d) \( f_{A \Delta B}(x) = 1 \iff x \in A \Delta B \iff (x \in A \) or \( x \in B \) or \( x \notin A \) and \( x \notin B \) \( \iff f_A(x) + f_B(x) - 2f_A(x)f_B(x) = 1 \)

59. a) Domain is \( Z \); codomain is \( R \); domain of definition is the set of nonzero integers; the set of values for which \( f \) is undefined is \{0\}; not a total function.
   b) Domain is \( Z \); codomain is \( Z \); domain of definition is \( Z \); set of values for which \( f \) is undefined is \( \{0\} \); total function.
   c) Domain is \( \mathbb{Z} \times \mathbb{Z} \); codomain is \( Z \); domain of definition is \( \mathbb{Z} \times \{0\} \); set of values for which \( f \) is undefined is \( \{0\} \); not a total function.
   d) Domain is \( \mathbb{Z} \times \mathbb{Z} \); codomain is \( Z \); domain of definition is \( \mathbb{Z} \times Z \); set of values for which \( f \) is undefined is \( \{0\} \); total function.
   e) Domain is \( \mathbb{Z} \times \mathbb{Z} \); codomain is \( Z \); domain of definition is \( \{(m, n) \mid m \geq n\} \); set of values for which \( f \) is undefined is \( \{(m, n) \mid m \leq n\} \); not a total function.

SECTION 1.7

1. a) 3  b) 1
   c) 787  d) 2639

3. a) \( a_0 = 2, a_1 = 3, a_2 = 5, a_3 = 9 \)
   b) \( a_0 = 1, a_1 = 4, a_2 = 27, a_3 = 256 \)

5. a) 2, 5, 8, 11, 14, 17, 20, 23, 26, 29
   b) 1, 1, 1, 2, 2, 2, 3, 3, 4
   c) 1, 1, 1, 3, 3, 5, 7, 7, 9, 9
   d) \(-1, -2, -2, 8, 8, 656, 4912, 40064, 362368, 3627776\)
   e) 3, 6, 12, 24, 48, 96, 192, 384, 768, 1536
   f) 1, 1, 2, 3, 5, 8, 13, 21, 34, 55
   g) 1, 2, 2, 3, 3, 2, 3, 4, 4, 4
   h) 3, 3, 5, 4, 4, 3, 5, 5, 4, 3

7. Each term could be twice the previous term; the \( n \)th term could be obtained from the previous term by adding \( n - 1 \); the terms could be the positive integers that are not multiples of 3; there are infinitely many other possibilities.

9. a) one 1 and one 0, followed by two 1s and two 0s, followed by three 1s and three 0s, and so on
   b) the positive integers are listed in increasing order with each even positive integer listed twice.
   c) the terms in odd-numbered locations are the successive powers of 2; the terms in even-numbered locations are all 0.
   d) \( a_n = 3 \cdot 2^{n-1} \)
   e) \( a_n = 15 - 7(n - 1) = 22 - 7n \)
   f) \( a_n = (n^2 + n + 4)/2 \)
   g) \( a_n = 2n^2 \)
   h) \( a_n = n! + 1 \)

11. Among the integers 1, 2, ..., \( a_n \), where \( a_n \) is the \( n \)th positive integer not a perfect square, the nonsquares are \( a_1, a_2, ..., a_n \) and the squares are \( 1^2, 2^2, ..., k^2 \), where \( k \) is the integer with \( k^2 < n + k < (k + 1)^2 \). Consequently, \( a_n = n + k \), where \( k^2 < a_n < (k + 1)^2 \). To find \( k \), first note that \( k^2 < n + k < (k + 1)^2 \), so \( k^2 + 1 \leq n + k \leq (k + 1)^2 - 1 \). Hence \( k^2 + 1 \leq k^2 + 1 \leq n \leq k^2 + k = (k + 1)^2 - 1 \). It follows that \( k - \frac{1}{2} < \sqrt{n} < k + \frac{1}{2} \), so \( k = \lfloor \sqrt{n} \rfloor \) and \( a_n = n + k = n + \lfloor \sqrt{n} \rfloor \).

13. a) 20  b) 11
   c) 30  d) 511

15. a) 1533  b) 510
   c) 4923  d) 9842

17. a) 21  b) 78
   c) 18  d) 18

19. \( \sum_{j=1}^n (a_j - a_{j-1}) = a_n - a_0 \)

21. a) \( n^2 \)  b) \( n(n + 1)/2 \)

23. 15150

25. \( \frac{3}{2}(n(n+1)(2n+1)) + \frac{n(n+1)}{2} + (n+1)(m-(n+1)^2+1) \),
where \( n = \lfloor \sqrt{m} \rfloor - 1 \)

27. a) 0  b) 1680
   c) 1  d) 1024

29. 34

31. a) countable, \(-1, -2, -3, -4, \ldots \)
   b) countable, 0, 2, -2, 4, -4, \ldots